# Multiplication in the cohomology of Grassmannians via Gröbner bases

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**Abstract**

In order to gain better understanding of the multiplication in the integral cohomology of the complex Grassmann manifold $G_{k,n}(\mathbb{C})$ (in the Borel's picture) a minimal strong Gröbner basis for the ideal $I_{k,n}$ determining this cohomology is obtained. These results are applied to obtain recurrence relations among Kostka numbers which completely determine these numbers. Corresponding results for real Grassmann manifolds are also presented.

*Key words:* Cohomology of Grassmannians, Gröbner bases, Kostka numbers, Chern classes, Stiefel-Whitney classes, Schubert classes, Pieri's formula
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## 1. Introduction

In this paper, we denote by $\mathbb{N}_0$ the set of nonnegative integers and by $\mathbb{N}$ the set of positive integers. Let $k, n \in \mathbb{N}$. The complex (resp. real) Grassmann manifold $G_{k,n}(\mathbb{C})$ (resp. $G_{k,n}(\mathbb{R})$) is the set of all $k$-dimensional subspaces of the vector space $\mathbb{C}^{n+k}$ (resp. $\mathbb{R}^{n+k}$), with the manifold structure coming from the natural identification $G_{k,n}(\mathbb{C}) = U(n+k)/U(n) \times U(k)$ (resp. $G_{k,n}(\mathbb{R}) = O(n+k)/O(n) \times O(k)$). In this paper we study the $\mathbb{Z}$-cohomology (resp. $\mathbb{Z}_2$-cohomology) of $G_{k,n}(\mathbb{C})$ (resp. $G_{k,n}(\mathbb{R})$).

There are several ways to describe $H^*(G_{k,n}(\mathbb{C}); \mathbb{Z})$ – the most notable are the ones using Schubert classes and using Chern classes of the canonical vector bundle over $G_{k,n}(\mathbb{C})$ (Borel's description from [2]). Both of these descriptions allow us to easily obtain an additive basis for $H^*(G_{k,n}(\mathbb{C}); \mathbb{Z})$. The first will be denoted by $\Sigma_{k,n}$ and the second by $B_{k,n}$ (see Section 2). So, in order to understand further the multiplicative structure of $H^*(G_{k,n}(\mathbb{C}); \mathbb{Z})$, it is of interest to obtain formulas that can be used to represent products of elements of $\Sigma_{k,n}$ (resp. $B_{k,n}$) in the basis $\Sigma_{k,n}$ (resp. $B_{k,n}$). It is well-known that in $\Sigma_{k,n}$ this can be done using Pieri's formula. One of the goal of this paper is to get a better understanding of $H^*(G_{k,n}(\mathbb{C}); \mathbb{Z})$, by studying it through the additive basis $B_{k,n}$. The change from basis $\Sigma_{k,n}$ to $B_{k,n}$ is established by a Kostka matrix, whose elements are hard to compute (see [11]). Therefore, in order to perform concrete calculations in $B_{k,n}$, we cannot rely only on the calculation in the basis $\Sigma_{k,n}$, but we need to develop specific techniques for calculating in $B_{k,n}$. In this paper, this is done by constructing (suitable) Gröbner bases for the ideals that, by Borel's description, determine $H^*(G_{k,n}(\mathbb{C}); \mathbb{Z})$.

By Borel's description, the $\mathbb{Z}_2$-cohomology of Grassmannian $G_{k,n}(\mathbb{R})$ is a polynomial algebra in Stiefel-Whitney classes of the canonical vector bundle over $G_{k,n}(\mathbb{R})$ modulo certain ideal. Although the description of this ideal is simple enough, concrete calculations in the cohomology of real Grassmannians may be rather

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difficult to perform. In various applications it is important to determine if a certain cohomology class, given in terms of Stiefel-Whitney classes, is zero or not — for example, in determining the span of Grassmannians, in discussing immersions and embeddings in Euclidean spaces, in the determination of cup-length (which is related to the Lusternik-Schnirelmann category), in some geometrical problems which may be reduced to the question of the existence of a non-zero section of a bundle over a Grassmann manifold, etc. It is evident from [6], [9] and [12], that having a Gröbner basis for the ideal that determines $H^*(G_{k,n}(\mathbb{R});\mathbb{Z}_2)$, can be very helpful for answering these kind of questions. In this paper, using the corresponding result for complex Grassmannians, we obtain Gröbner bases for these ideals.

The paper is organized in the following way. In Section 2 we give a brief overview of the theory of symmetric functions and the $\mathbb{Z}$-cohomology of complex Grassmannian (that is relevant to our work), as well as some definitions and basic propositions of the theory of Gröbner bases. In Section 3 we construct (minimal) strong Gröbner bases for the ideals determining $H^*(G_{k,n}(\mathbb{C});\mathbb{Z})$. This extends the results obtained in [13], where Gröbner bases were obtained for $k \leq 3$. As an immediate consequence of these results, in Section 4 we derive a multiplication formula for elements of the additive basis $B_{k,n}$ of $H^*(G_{k,n}(\mathbb{C});\mathbb{Z})$ (formula (19)), which can be understood as the analogue of the Pieri's formula. In Section 5 we use the results from Section 3, and obtain a recurrence formula that (uniquely) determines Kostka numbers. In [4] and [5] recurrence formulas of similar flavour were obtained for inverse Kostka matrix. Finally, in Section 6, using the results for the complex Grassmannians, we construct Gröbner bases for the ideals that, by Borel's description, determine the $\mathbb{Z}_2$-cohomology of real Grassmannians, thus completing the research done in [9] and [12].

## 2. Preliminaries

### *2.1. Symmetric functions and $\mathbb{Z}$-cohomology of complex Grassmannians*

A *partition* $\lambda = (l_1, l_2, \ldots, l_s)$ is a non-increasing sequence of integers $l_1 \geq l_2 \geq \cdots \geq l_s \geq 0$. Furthermore, $l(\lambda) := \max\{t : l_t \neq 0\}$ is the *length* and $|\lambda| := l_1 + l_2 + \cdots + l_s$ is the *weight* of the partition $\lambda$. For $\lambda$ such that $l_1 \leq n$ and $l(\lambda) \leq k$ we will write $\lambda \subset k \times n$, since its Young diagram is contained in the rectangle $k \times n$ (see Figure 1). We will say that partitions $(l_1, l_2, \ldots, l_s)$ and $(l_1, l_2, \ldots, l_s, \underbrace{0, \ldots, 0}_{k})$ are the same (for any $k$). This allows us to introduce addition in the set of all partitions, as addition of the appropriate vectors.

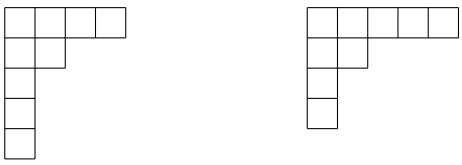

**Figure 1.** Young diagram of partition $\lambda = (4, 2, 1, 1, 1)$ (on the left) and partition $\lambda^* = (5, 2, 1, 1)$ (on the right).

In the set of all partitions we can introduce a partial order $\geq$ (called the *dominance order*) in the following way: for partitions $\lambda = (l_1, \ldots, l_s)$ and $\mu = (m_1, \ldots, m_s)$ one has $\lambda \geq \mu$ if $|\lambda| = |\mu|$ and

$$l_1 + l_2 + \cdots + l_i \geq m_1 + m_2 + \cdots + m_i$$

for all $i$ such that $1 \leq i \leq s$.

For $\lambda \subset k \times n$ its *conjugate partition* $\lambda^* \subset n \times k$ is the partition obtained from $\lambda$ by diagonal symmetry.

For a partition $\lambda$ and a vector $\mu$ with nonnegative integer components, the *Kostka number* $K_{\lambda\mu}$ is defined as the number of semistandard Young tableaux with shape $\lambda$ and type $\mu$ (see [14, p. 311]). Apart from obvious combinatorial interest in these numbers, they play a prominent role in other fields of mathematics: representation theory, topology, geometry, etc. In this paper we will study these numbers through the theory of symmetric functions and cohomology of Grassmannians.

Let $\Lambda_k$ be the ring of symmetric functions in the variables $x_1, x_2, \ldots, x_k$. We recall some $\mathbb{Z}$-bases of $\Lambda_k$ which are going to be important for us.

For a $k$-tuple $\alpha = (a_1, \dots, a_k)$ of nonnegative integers, let $a_\alpha = \det(x_j^{a_i})_{k\times k}$. *Schur's function* associated to a partition $\lambda$ of length at most $k$, denoted by $s_\lambda$, is defined as $a_{\lambda+\delta}/a_\delta$, where $\delta = (k-1, k-2, \dots, 0)$.

For $1 \le m \le k$, let $e_m \in \Lambda_k$ (resp. $h_m \in \Lambda_k$) denote the *elementary* (resp. *complete*) symmetric function of degree $m$. The following identities hold: $e_m = s_{(1^m)}$ and $h_m = s_{(m)}$, where $(1^m)$ for every $m \le k$ denotes the partition consisting of $m$ ones (see [8, p. 10]).

Additionally, for a partition $\lambda = (l_1, l_2, \dots, l_s)$ such that $l_1 \le k$, let $e_\lambda = e_{l_1} e_{l_2} \cdots e_{l_s}$ (resp. $h_\lambda = h_{l_1} h_{l_2} \cdots h_{l_s}$). The following is well-known (see [8], [14]).

**Proposition 1.** *Each of the following sets forms a $\mathbb{Z}$-basis for $\Lambda_k$:*

*1)* $\{s_\lambda : \lambda$ *is a partition such that* $l(\lambda) \le k\}$*;*

*2)* $\{e_\lambda : \lambda$ *is a partition such that* $l_1 \le k\}$*;*

*3)* $\{h_\lambda : \lambda$ *is a partition such that* $l_1 \le k\}$*.*

The transition between these bases is achieved by Kostka numbers, i.e., for a partition $\mu$ one has (see [14, Corollary 7.12.4] and [14, p. 335])

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda, \quad e_\mu = \sum_\lambda K_{\lambda\mu} s_{\lambda^*}, \tag{1}$$

where the sums are over all partitions $\lambda$, $|\lambda| = |\mu|$, that satisfy $l(\lambda) \le k$ for the first one, and $l_1 \le k$ for the second. Since $K_{\lambda\lambda} = 1$ and $K_{\lambda\mu} \ne 0$ only if $\lambda \ge \mu$ (see [14, Proposition 7.10.5]), if we extend dominance order to any linear order on the set of all partitions, the matrix $(K_{\lambda\mu})$ is lower triangular with ones on the main diagonal.

Let $n, k \in \mathbb{N}$, $V := \mathbb{C}^{n+k}$ and a complete flag

$$0 = V_0 \subset V_1 \subset \cdots \subset V_i \subset \cdots \subset V_{n+k} = V.$$

For a partition $\lambda \subset k \times n$, $\lambda = (l_1, l_2, \dots, l_k)$, observe the associated *Schubert variety*

$$X_\lambda = \{W \in G_{k,n} : \dim(W \cap V_{n+i-l_i}) \ge i, 1 \le i \le k\}.$$

Denote by $\sigma_\lambda$ the class of $X_\lambda$ in the cohomology ring $H^*(G_{k,n}(\mathbb{C}); \mathbb{Z})$. Then $\Sigma_{k,n} := \{\sigma_\lambda : \lambda \subset k \times n\}$ forms an additive basis of this ring (see [8]). Moreover, the multiplication in this ring is determined by the *Pieri's formula*

$$\sigma_\lambda \cdot \sigma_{(m)} = \sum_\nu \sigma_\nu,$$

where the sum is over all partitions $\nu$ which can be obtained by adding $m$ boxes to Young's diagram of $\lambda$ with no two in the same column. This formula leads to a surjective morphism of rings

$$\phi_{k,n} : \Lambda_k \to H^*(G_{k,n}(\mathbb{C}); \mathbb{Z}),$$

given by $s_\lambda \mapsto \sigma_\lambda$ for $\lambda \subset k \times n$, and $s_\lambda \mapsto 0$ for $\lambda \not\subset k \times n$ (see [8, Corollary 3.2.9]).

One other standard presentation of the cohomology ring of the Grassmannians is due to Borel. By this description

$$H^*(G_{k,n}(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \dots, c_k]/I_{k,n},$$

where $c_1, c_2, \dots, c_k$ are the Chern classes of the canonical complex vector bundle $\gamma_k$ over $G_{k,n}$, and $I_{k,n} = (\overline{c}_{n+1}, \overline{c}_{n+2}, \dots, \overline{c}_{n+k})$ is the ideal generated by dual classes. These dual classes satisfy

$$(1 + c_1 + c_2 + \cdots + c_k)(1 + \overline{c}_1 + \overline{c}_2 + \cdots) = 1,$$

which leads to

$$\overline{c}_{m+k} = -\sum_{i=1}^{k} c_i \overline{c}_{m+k-i}, \quad \text{for } m \ge 1, \tag{2}$$

and

$$\overline{c}_r = \sum_{a_1+2a_2+\cdots+ka_k=r} (-1)^{a_1+a_2+\cdots+a_k}[a_1,a_2,\ldots,a_k]c_1^{a_1}c_2^{a_2}\cdots c_k^{a_k}, \quad r \geq 1 \tag{3}$$

where $[a_1,a_2,\ldots,a_k] := \binom{a_1+a_2+\cdots+a_k}{a_1}\binom{a_2+\cdots+a_k}{a_2}\cdots\binom{a_{k-1}+a_k}{a_{k-1}}$ is the multinomial coefficient.

It is well-known that identities

$$\sigma_{(1^i)} = (-1)^i c_i, \quad 1 \leq i \leq k, \qquad \text{and} \qquad \sigma_{(i)} = \overline{c}_i, \quad 1 \leq i \leq n,$$

hold in $H^*(G_{k,n}(\mathbb{C});\mathbb{Z})$. From the first one, in view of (1) and the morphism $\phi_{k,n}$, we obtain the formula

$$c_{m_1}c_{m_2}\cdots c_{m_s} = (-1)^{|\mu|}\sum_{\lambda} K_{\lambda\mu}\sigma_{\lambda^*}, \tag{4}$$

where $\mu = (m_1,m_2,\ldots,m_s)$ is a partition such that $m_1 \leq k$, and the sum is over all partitions $\lambda \subset n \times k$, such that $|\lambda| = |\mu|$. Particularly, if we restrict attention to partitions $\mu = (m_1,m_2,\ldots,m_s)$ such that $\mu \subset n \times k$ (i.e., $s \leq n$ and $m_1 \leq k$), using (4), the fact that $\Sigma_{k,n}$ is an additive basis for $H^*(G_{k,n}(\mathbb{C});\mathbb{Z})$ and the fact that the Kostka matrix $(K_{\lambda\mu})$ is lower triangular with ones on the main diagonal, we obtain the following result.

**Proposition 2.** *The set*

$$B_{k,n} := \{c_{m_1}c_{m_2}\cdots c_{m_s} \,:\, s \leq n,\, 1 \leq m_s \leq \cdots \leq m_1 \leq k\} = \{c_1^{a_1}c_2^{a_2}\cdots c_k^{a_k} \,:\, a_1+a_2+\cdots+a_k \leq n\}$$

*is an additive basis for* $H^*(G_{k,n}(\mathbb{C});\mathbb{Z})$.

Finally, we note that, when calculating Kostka number $K_{\lambda\mu}$, we may assume that $\mu$ is a partition. This is a consequence of the well-known fact that $K_{\lambda\mu}$ is invariant under permutations of coordinates of the vector $\mu$.

*2.2. Gröbner bases*

Let $R$ be a principal ideal domain and $R[x_1,x_2,\ldots,x_k]$ the polynomial algebra over $R$ in $k$ variables. A *monomial* on variables $x_1,x_2,\ldots,x_k$ is a product $x_1^{a_1}x_2^{a_2}\cdots x_k^{a_k} \in R[x_1,x_2,\ldots,x_k]$, where $a_1,a_2,\ldots,a_k \in \mathbb{N}_0$. The set of all monomials in $R[x_1,x_2,\ldots,x_k]$ will be denoted by $M$. A *term* in $R[x_1,x_2,\ldots,x_k]$ is a product $\alpha m$, where $\alpha \in R$ and $m \in M$.

Let $\preceq$ be a well ordering of $M$ (a total ordering such that every nonempty subset of $M$ has a least element) with the property that $m_1 \preceq m_2$ implies $mm_1 \preceq mm_2$, for all $m,m_1,m_2 \in M$.

For a polynomial $f = \sum_{i=1}^r \alpha_i m_i \in R[x_1,x_2,\ldots,x_k]$, such that $\alpha_i \in R\setminus\{0\}$, and $m_i \in M$ are pairwise different, let $M(f) := \{m_i \,|\, 1 \leq i \leq r\}$. We define the *leading monomial of* $f$, denoted by $\mathrm{LM}(f)$, as $\max M(f)$ with respect to $\preceq$. The *leading coefficient of* $f$, denoted by $\mathrm{LC}(f)$, is the coefficient of $\mathrm{LM}(f)$ and the *leading term of* $f$ is $\mathrm{LT}(f)$:=$\mathrm{LC}(f)\cdot\mathrm{LM}(f)$.

Strong Gröbner basis of an ideal $I$ of $R[x_1,x_2,\ldots,x_k]$ can be defined in many equivalent ways. In this paper, we will define it as it was done in [1].

**Definition 1.** Let $G \subset R[x_1,x_2,\ldots,x_k]$ be a finite set of non-zero polynomials and $I_G = (G)$ the ideal in $R[x_1,x_2,\ldots,x_k]$ generated by $G$. We say that $G$ is a *strong Gröbner basis* for $I_G$ (with respect to $\preceq$) if for each $f \in I_G\setminus\{0\}$ there exists $g \in G$ such that $\mathrm{LT}(g) \mid \mathrm{LT}(f)$, i.e., $\mathrm{LT}(f) = t\cdot\mathrm{LT}(g)$, for some term $t$. Strong Gröbner basis $G$ is *minimal* if $\mathrm{LT}(g') \nmid \mathrm{LT}(g'')$, for all distinct $g',g'' \in G$.

**Remark 1.** If $R$ is a field, then a strong Gröbner basis (from Definition 1) is simply called *Gröbner basis*. Additionally, for a finite subset $G$ of $R[x_1,x_2,\ldots,x_k]\setminus\{0\}$ we say that it is a *reduced Gröbner basis* of $I_G := (G)$ if:

(i) $G$ is a Gröbner basis of $I_G$;

(ii) $\mathrm{LC}(g) = 1$ for all $g \in G$;

(iii) $\mathrm{LT}(g')$ does not divide any monomial from $M(g'')$, for all distinct $g',g'' \in G$.

Reduced Gröbner bases are quite important in the theory of Gröbner bases over a field – they are optimal in certain sense and they are unique for a fixed monomial ordering (see [1]).

## 3. Gröbner bases for complex Grassmannians

Recall that for $\alpha, \beta \in \mathbb{Z}$ the binomial coefficient $\binom{\alpha}{\beta}$ is defined by

$$\binom{\alpha}{\beta} := \begin{cases} \dfrac{\alpha(\alpha-1)\cdots(\alpha-\beta+1)}{\beta!}, & \beta > 0 \\ 1, & \beta = 0 \\ 0, & \beta < 0 \end{cases},$$

and therefore, the following lemma is straightforward.

**Lemma 3.** *If $\binom{\alpha}{\beta} \neq 0$, then $\alpha \geq \beta$ or $\alpha \leq -1$.*

Recall also the well-known formula (which holds for all $\alpha, \beta \in \mathbb{Z}$)

$$\binom{\alpha}{\beta} = \binom{\alpha-1}{\beta} + \binom{\alpha-1}{\beta-1}. \tag{5}$$

Let us now introduce some notations that we are going to use throughout this paper. Let $m \in \mathbb{N}$ and $f_1 = (1, 0, \ldots, 0)$, $f_2 = (0, 1, 0, \ldots, 0)$, $\ldots$ , $f_m = (0, \ldots, 0, 1)$ be the vectors of the standard basis for $\mathbb{Z}^m$ and we put $f_0 = (0, 0, \ldots, 0) \in \mathbb{Z}^m$. For an $m$-tuple $\lambda$ of integers we define the following $m$-tuples obtained from $\lambda$ (for $0 \leq i \leq j \leq m$):

- $\lambda^i = \lambda + f_i$ and $\lambda_i = \lambda - f_i$;
- $\lambda^{i,j} = \lambda + f_i + f_j$ and $\lambda_{i,j} = \lambda - f_i - f_j$.

For $k \geq 2$, a $k$-tuple $\alpha = (a_1, a_2, \ldots, a_k)$ and a $(k-1)$-tuple $\mu = (m_2, \ldots, m_k)$ of integers, let:

- $|\alpha| := \sum_{j=1}^{k} a_j$, $\|\alpha\| := \sum_{j=1}^{k} j a_j$, and $|\mu| := \sum_{j=2}^{k} m_j$, $\|\mu\| := \sum_{j=2}^{k} (j-1) m_j$;
- $[\alpha, \mu]_t := \binom{\sum_{j=t-1}^{k} a_j - \sum_{j=t}^{k} m_j}{a_{t-1}}$, $\quad 2 \leq t \leq k$;
- $[\alpha, \mu] := \prod_{t=2}^{k} [\alpha, \mu]_t$.

For example, $[\alpha, \mu]_2 = \binom{|\alpha| - |\mu|}{a_1}$. Also, $[\alpha, \mathbf{0}] = [a_1, a_2, \ldots, a_k]$, where $\mathbf{0} = (\underbrace{0, \ldots, 0}_{k-1})$.

**Remark 2.** The case $k = 1$ will be allowed as well. Then $\mu$ must be $\emptyset$, $|\mu| = \|\mu\| = 0$, $[\alpha, \mu] = 1$ for any $\alpha = (a_1)$.

**Remark 3.** Note that the $(k-1)$-tuple $\mu = (m_2, \ldots, m_k)$ is indexed by integers from 2 to $k$, not from 1 to $k-1$. The reason for this becomes clear in Proposition 5.

Henceforth, the integers $k, n \in \mathbb{N}$ are fixed. Observe the polynomial algebra $\mathbb{Z}[c_1, c_2, \ldots, c_k]$. For a $k$-tuple $\alpha = (a_1, a_2, \ldots, a_k)$ of nonnegative integers, the monomial $c_1^{a_1} c_2^{a_2} \cdots c_k^{a_k}$ will be abbreviated to $C^\alpha$. Let us now define certain polynomials in $\mathbb{Z}[c_1, c_2, \ldots, c_k]$ which will be important in our considerations.

**Definition 2.** For a $(k-1)$-tuple of nonnegative integers $\mu = (m_2, \ldots, m_k)$, let

$$g_\mu := \sum_{\|\alpha\|=n+1+\|\mu\|} (-1)^{n+1+|\alpha|}[\alpha, \mu] \cdot C^\alpha,$$

where the sum is taken over all $k$-tuples of nonnegative integers $\alpha = (a_1, a_2, \ldots, a_k)$ such that $\|\alpha\| = n + 1 + \|\mu\|$.

Moreover, let

$$G^{\mathbb{C}} := \{g_\mu \,:\, |\mu| \leq n+1\}.$$

Note that, by (3), $\overline{c}_{n+1} = (-1)^{n+1} g_{\mathbf{0}} \in G^{\mathbb{C}}$.

Our aim is to prove that $G^{\mathbb{C}}$ is a strong Gröbner basis for the ideal $I_{k,n} = (\overline{c}_{n+1}, \overline{c}_{n+2}, \ldots, \overline{c}_{n+k})$ which determines the cohomology algebra $H^*(G_{k,n}(\mathbb{C}); \mathbb{Z})$. In order to do so, first we need to specify a monomial ordering in $\mathbb{Z}[c_1, c_2, \ldots, c_k]$. We shall use the grlex ordering $\preccurlyeq$ on the monomials in $\mathbb{Z}[c_1, c_2, \ldots, c_k]$ with $c_1 > c_2 > \cdots > c_k$. It is defined as follows. For $k$-tuples $\alpha = (a_1, a_2, \ldots, a_k)$ and $\beta = (b_1, b_2, \ldots, b_k)$, $C^\alpha \prec C^\beta$ if and only if one of the following holds:

1) $|\alpha| < |\beta|$;

2) $|\alpha| = |\beta|$ and $a_s < b_s$, where $s = \min\{i \,:\, a_i \neq b_i\}$.

As usual, $C^\alpha \preccurlyeq C^\beta$ means that either $C^\alpha \prec C^\beta$ or $C^\alpha = C^\beta$.

**Lemma 4.** *If a $k$-tuple $\alpha = (a_1, \ldots, a_k)$ and a $(k-1)$-tuple $\mu = (m_2, \ldots, m_k)$ of nonnegative integers are such that $[\alpha, \mu] \neq 0$, then $|\alpha| < |\mu|$ or else $\sum_{j=t}^k a_j \geq \sum_{j=t}^k m_j$ for all $t$ such that $2 \leq t \leq k$.*

Proof. Let us assume that $|\alpha| \geq |\mu|$. Using mathematical induction on $t$, we will prove that $\sum_{j=t}^k a_j \geq \sum_{j=t}^k m_j$, for $2 \leq t \leq k$. Since

$$\binom{|\alpha| - |\mu|}{a_1} = [\alpha, \mu]_2 \neq 0 \quad \text{and} \quad |\alpha| - |\mu| \geq 0,$$

by Lemma 3 we have that $|\alpha| - |\mu| \geq a_1$, and therefore $\sum_{j=2}^k a_j \geq \sum_{j=2}^k m_j$.

Suppose now that $\sum_{j=t}^k a_j \geq \sum_{j=t}^k m_j$, for some $t$ such that $2 \leq t \leq k-1$. Since $[\alpha, \mu]_{t+1} \neq 0$ and $\sum_{j=t}^k a_j \geq \sum_{j=t}^k m_j \geq \sum_{j=t+1}^k m_j$, again by Lemma 3 we conclude that $\sum_{j=t}^k a_j - \sum_{j=t+1}^k m_j \geq a_t$. Hence, $\sum_{j=t+1}^k a_j \geq \sum_{j=t+1}^k m_j$. □

**Proposition 5.** *Let $\mu = (m_2, \ldots, m_k)$ be a $(k-1)$-tuple of nonnegative integers such that $|\mu| \leq n+1$ (i.e., such that $g_\mu \in G^{\mathbb{C}}$). Then $g_\mu \neq 0$ and $\mathrm{LT}(g_\mu) = C^{\overline{\mu}}$, where $\overline{\mu} = (n+1-|\mu|, m_2, \ldots, m_k)$. Moreover, if $C^\alpha \in M(g_\mu) \setminus \{C^{\overline{\mu}}\}$, for some $k$-tuple $\alpha$ of nonnegative integers, then $|\alpha| < n+1$.*

Proof. If we put $m_1 = n+1-|\mu|$, then obviously $[\overline{\mu}, \mu]_t = \binom{m_{t-1}}{m_{t-1}} = 1$, for all $t$ such that $2 \leq t \leq k$, and therefore $[\overline{\mu}, \mu] = 1$. Furthermore,

$$\|\overline{\mu}\| = \sum_{j=1}^k j m_j = n+1-|\mu| + \sum_{j=2}^k j m_j = n+1 + \sum_{j=2}^k (j-1) m_j = n+1+\|\mu\|.$$

Hence, $C^{\overline{\mu}} \in M(g_\mu)$ and the coefficient of $C^{\overline{\mu}}$ in $g_\mu$ is 1. So, $g_\mu \neq 0$.

Now take a $k$-tuple $\alpha = (a_1, \ldots, a_k)$ of nonnegative integers such that $\|\alpha\| = n+1+\|\mu\|$ and $[\alpha, \mu] \neq 0$, i.e., $C^\alpha \in M(g_\mu)$. Since $|\overline{\mu}| = n+1$, in order to finish the proof of the proposition it suffices to show that if $|\alpha| \geq n+1$ then $\alpha = \overline{\mu}$.

Since $|\mu| \leq n+1 \leq |\alpha|$, by Lemma 4 we have the following $k-1$ inequalities:

$$\begin{array}{rcl} a_k & \geq & m_k, \\ a_{k-1}+a_k & \geq & m_{k-1}+m_k, \\ & \vdots & \\ a_2+\cdots+a_k & \geq & m_2+\cdots+m_k. \end{array} \tag{6}$$

Summing up these inequalities we get

$$\sum_{j=2}^{k}(j-1)a_j \geq \sum_{j=2}^{k}(j-1)m_j.$$

On the other hand, since $|\alpha| \geq n+1$ and $\|\alpha\| = n+1+\|\mu\|$,

$$\sum_{j=2}^{k}(j-1)a_j = \sum_{j=1}^{k}(j-1)a_j = \|\alpha\| - |\alpha| \leq \|\alpha\| - (n+1) = \|\mu\| = \sum_{j=2}^{k}(j-1)m_j,$$

so all the inequalities in (6) are in fact equalities, and $|\alpha| = n+1$. Therefore, $a_t = m_t$ for $2 \leq t \leq k$, and $a_1 = |\alpha| - \sum_{j=2}^{k} a_j = n+1-|\mu|$, i.e., $\alpha = \overline{\mu}$. □

Prior to the formulation of the following lemma, we would like to emphasize that for a $(k-1)$-tuple $\mu = (m_2, \ldots, m_k)$, by our definition, $\mu^i = (m_2, \ldots, m_{i+1}+1, \ldots, m_k)$, $1 \leq i \leq k-1$, and likewise for $\mu^{i,j}$, $\mu_i$ and $\mu_{i,j}$. For example, the $(k-1)$-tuple $\mu^2$ is defined as $(m_2, m_3+1, \ldots, m_k)$, and not as $(m_2+1, m_3, \ldots, m_k)$.

**Lemma 6.** *Let $\alpha = (a_1, a_2, \ldots, a_k)$ be a $k$-tuple and $\mu = (m_2, \ldots, m_k)$ a $(k-1)$-tuple of integers.*

*a) For $1 \leq i \leq j \leq k-2$ we have*

$$[\alpha, \mu^{i,j}] = -[\alpha_i, \mu^j] + [\alpha, \mu^{i-1,j+1}] + [\alpha_{j+1}, \mu^{i-1}].$$

*b) For $1 \leq i \leq k-1$ we have*

$$[\alpha, \mu^{i,k-1}] = -[\alpha_i, \mu^{k-1}] + [\alpha_k, \mu^{i-1}].$$

Proof. Let $1 \leq i \leq j \leq k-1$. It is immediate from the definition that for all $t$ such that $2 \leq t \leq k$

$$[\alpha, \mu^{i,j}]_t = \binom{a_{t-1}+a_t+\cdots+a_k-m_t-\cdots-m_k-\delta_t}{a_{t-1}},$$

where $\delta_t = \begin{cases} 2, & t \leq i+1 \\ 1, & i+2 \leq t \leq j+1 \\ 0, & t > j+1 \end{cases}$. Also, if $t \neq i+1$, then

$$[\alpha_i, \mu^j]_t = \binom{a_{t-1}+a_t+\cdots+a_k-m_t-\cdots-m_k-\delta_t}{a_{t-1}},$$

and so,

$$[\alpha, \mu^{i,j}]_t = [\alpha_i, \mu^j]_t, \qquad \text{for } t \neq i+1. \tag{7}$$

Likewise, using formula (5) we get

$$[\alpha, \mu^{i,j}]_{i+1} + [\alpha_i, \mu^j]_{i+1} = [\alpha, \mu^j]_{i+1}, \tag{8}$$

since the left-hand side is equal to

$$\binom{a_i+\cdots+a_k-m_{i+1}-\cdots-m_k-2}{a_i} + \binom{a_i+\cdots+a_k-m_{i+1}-\cdots-m_k-2}{a_i-1},$$

and right-hand side to

$$\binom{a_i + \cdots + a_k - m_{i+1} - \cdots - m_k - 1}{a_i}.$$

a) In this case, similarly as for (7) and (8), one obtains the following equalities:

$$[\alpha_i, \mu^j]_t = [\alpha, \mu^{i-1,j+1}]_t, \quad \text{for } t \notin \{i+1, j+2\} \tag{9}$$

$$[\alpha, \mu^{i-1,j+1}]_t = [\alpha_{j+1}, \mu^{i-1}]_t, \quad \text{for } t \neq j+2 \tag{10}$$

$$[\alpha, \mu^j]_{i+1} = [\alpha, \mu^{i-1,j+1}]_{i+1} \tag{11}$$

$$[\alpha_i, \mu^j]_{j+2} = [\alpha, \mu^{i-1,j+1}]_{j+2} + [\alpha_{j+1}, \mu^{i-1}]_{j+2}. \tag{12}$$

So, using identities (7)–(12), we have

$$\begin{aligned}
[\alpha, \mu^{i,j}] &= \prod_{t=2}^{k} [\alpha, \mu^{i,j}]_t = [\alpha, \mu^{i,j}]_{i+1} \cdot \prod_{\substack{t=2 \\ t \neq i+1}}^{k} [\alpha_i, \mu^j]_t \\
&= \left(-[\alpha_i, \mu^j]_{i+1} + [\alpha, \mu^j]_{i+1}\right) \cdot \prod_{\substack{t=2 \\ t \neq i+1}}^{k} [\alpha_i, \mu^j]_t \\
&= -\prod_{t=2}^{k} [\alpha_i, \mu^j]_t + [\alpha, \mu^{i-1,j+1}]_{i+1} \cdot \prod_{\substack{t=2 \\ t \neq i+1}}^{k} [\alpha_i, \mu^j]_t \\
&= -[\alpha_i, \mu^j] + [\alpha_i, \mu^j]_{j+2} \cdot \prod_{\substack{t=2 \\ t \neq j+2}}^{k} [\alpha, \mu^{i-1,j+1}]_t \\
&= -[\alpha_i, \mu^j] + \left([\alpha, \mu^{i-1,j+1}]_{j+2} + [\alpha_{j+1}, \mu^{i-1}]_{j+2}\right) \cdot \prod_{\substack{t=2 \\ t \neq j+2}}^{k} [\alpha, \mu^{i-1,j+1}]_t \\
&= -[\alpha_i, \mu^j] + [\alpha, \mu^{i-1,j+1}] + [\alpha_{j+1}, \mu^{i-1}].
\end{aligned}$$

b) In a similar manner as before, for $1 \leq i \leq k-1$, one can obtain two additional equalities:

$$[\alpha_i, \mu^{k-1}]_t = [\alpha_k, \mu^{i-1}]_t, \quad \text{for } t \neq i+1, \tag{13}$$

$$[\alpha, \mu^{k-1}]_{i+1} = [\alpha_k, \mu^{i-1}]_{i+1}. \tag{14}$$

Now, using identities (7), (8), (13) and (14), we have

$$\begin{aligned}
[\alpha, \mu^{i,k-1}] &= \prod_{t=2}^{k} [\alpha, \mu^{i,k-1}]_t = [\alpha, \mu^{i,k-1}]_{i+1} \cdot \prod_{\substack{t=2 \\ t \neq i+1}}^{k} [\alpha_i, \mu^{k-1}]_t \\
&= \left(-[\alpha_i, \mu^{k-1}]_{i+1} + [\alpha, \mu^{k-1}]_{i+1}\right) \cdot \prod_{\substack{t=2 \\ t \neq i+1}}^{k} [\alpha_i, \mu^{k-1}]_t \\
&= -[\alpha_i, \mu^{k-1}] + [\alpha_k, \mu^{i-1}],
\end{aligned}$$

and we are done. □

Note that we could unify parts a) and b) of the previous lemma by stating that

$$[\alpha, \mu^{i,j}] = -[\alpha_i, \mu^j] + [\alpha_{j+1}, \mu^{i-1}] + [\alpha, \mu^{i-1,j+1}], \text{ for } 1 \leq i \leq j \leq k-1,$$

with the convention that $[\alpha, \mu^{i-1,j+1}] = 0$ if $j = k-1$.

**Proposition 7.** *Let $\mu = (m_2, \dots, m_k)$ be a $(k-1)$-tuple of nonnegative integers and $1 \le i \le j \le k-1$. Then in the polynomial algebra $\mathbb{Z}[c_1, c_2, \dots, c_k]$, we have the following identity*

$$g_{\mu^{i,j}} = c_i g_{\mu^j} - c_{j+1} g_{\mu^{i-1}} + g_{\mu^{i-1,j+1}},$$

*where the polynomial $g_{\mu^{i-1,j+1}}$ is understood to be zero if $j = k-1$.*

Proof. By Lemma 6 we have

$$\begin{aligned} g_{\mu^{i,j}} &= \sum_{\|\alpha\|=n+1+\|\mu^{i,j}\|} (-1)^{n+1+|\alpha|}[\alpha, \mu^{i,j}] \cdot C^\alpha \\ &= \sum_{\|\alpha\|=n+1+\|\mu^{i,j}\|} (-1)^{n+1+|\alpha|} \left(-[\alpha_i, \mu^j] + [\alpha_{j+1}, \mu^{i-1}] + [\alpha, \mu^{i-1,j+1}]\right) \cdot C^\alpha \\ &= \sum_{\|\alpha\|=n+1+\|\mu^{i,j}\|} (-1)^{n+1+|\alpha_i|}[\alpha_i, \mu^j] \cdot C^\alpha - \sum_{\|\alpha\|=n+1+\|\mu^{i,j}\|} (-1)^{n+1+|\alpha_{j+1}|}[\alpha_{j+1}, \mu^{i-1}] \cdot C^\alpha \\ &\quad + g_{\mu^{i-1,j+1}}, \end{aligned}$$

since $\|\mu^{i,j}\| = \|\mu\| + i + j = \|\mu\| + i - 1 + j + 1 = \|\mu^{i-1,j+1}\|$ (for $j \le k-2$). Observe also that the equality $\|\alpha\| = n + 1 + \|\mu^{i,j}\|$ is equivalent to $\|\alpha_i\| = \|\alpha\| - i = n + 1 + \|\mu^{i,j}\| - i = n + 1 + \|\mu^j\|$, and likewise it is equivalent to $\|\alpha_{j+1}\| = n + 1 + \|\mu^{i-1}\|$.

Now, consider the first sum in the upper expression. Since the sum is taken over the $k$-tuples $\alpha = (a_1, a_2, \dots, a_k)$ of nonnegative integers (such that $\|\alpha\| = n + 1 + \|\mu^{i,j}\|$), the coordinates of $\alpha_i$ are also nonnegative with the exception that its $i$-th coordinate might be $-1$ (if $a_i = 0$). But, in that case,

$$[\alpha_i, \mu^j]_{i+1} = \binom{a_{i+1} + \cdots + a_k - m_{i+1} - \cdots - m_k - 2}{-1} = 0,$$

and so $[\alpha_i, \mu^j] = 0$. Therefore, we may assume that $a_i \ge 1$, and consequently, that $\alpha_i$ runs through the set of $k$-tuples of nonnegative integers (such that $\|\alpha_i\| = n + 1 + \|\mu^j\|$). Hence,

$$\sum_{\|\alpha\|=n+1+\|\mu^{i,j}\|} (-1)^{n+1+|\alpha_i|}[\alpha_i, \mu^j] \cdot C^\alpha = c_i \sum_{\|\alpha_i\|=n+1+\|\mu^j\|} (-1)^{n+1+|\alpha_i|}[\alpha_i, \mu^j] \cdot C^{\alpha_i} = c_i g_{\mu^j}.$$

So, we are left to prove that the second sum in the upper expression for $g_{\mu^{i,j}}$ is equal to $c_{j+1} g_{\mu^{i-1}}$. Let $\alpha = (a_1, a_2, \dots, a_k)$ be a $k$-tuple of nonnegative integers such that $\|\alpha\| = n + 1 + \|\mu^{i,j}\|$, i.e., $\|\alpha_{j+1}\| = n + 1 + \|\mu^{i-1}\|$. It suffices to show that $a_{j+1} = 0$ implies $[\alpha_{j+1}, \mu^{i-1}] = 0$, since then the proof follows as for the first sum.

If $j + 1 < k$, then $a_{j+1} = 0$ implies $[\alpha_{j+1}, \mu^{i-1}]_{j+2} = 0$, and therefore, $[\alpha_{j+1}, \mu^{i-1}] = 0$.

For $j = k-1$, let us assume to the contrary that $a_k = 0$ and $[\alpha_k, \mu^{i-1}] \ne 0$. First we shall prove that

$$a_{t-1} + a_t + \cdots + a_{k-1} \le m_t + \cdots + m_k + \varepsilon_t, \qquad \text{for all } t \in \{2, 3, \dots, k\}, \tag{15}$$

where $\varepsilon_t = \begin{cases} 1, & 2 \le t \le i \\ 0, & i+1 \le t \le k \end{cases}$. The proof is by reverse induction on $t$. For the induction base we prove (15) for $t = k$. Since $\binom{a_{k-1}-1-m_k}{a_{k-1}} = [\alpha_k, \mu^{i-1}]_k \ne 0$ and $a_{k-1} - 1 - m_k < a_{k-1}$, by Lemma 3 we conclude that $a_{k-1} - 1 - m_k \le -1$, so $a_{k-1} \le m_k = m_k + \varepsilon_k$. For the inductive step, let $2 \le t \le k-1$, and suppose that $a_t + \cdots + a_{k-1} \le m_{t+1} + \cdots + m_k + \varepsilon_{t+1}$. Since obviously $\varepsilon_{t+1} \le \varepsilon_t$, we actually have that $a_t + \cdots + a_{k-1} \le m_{t+1} + \cdots + m_k + \varepsilon_t$. Since

$$[\alpha_k, \mu^{i-1}]_t = \binom{a_{t-1} + a_t + \cdots + a_{k-1} - 1 - m_t - m_{t+1} - \cdots - m_k - \varepsilon_t}{a_{t-1}} \ne 0,$$

and $a_{t-1} + a_t + \cdots + a_{k-1} - 1 - m_t - m_{t+1} - \cdots - m_k - \varepsilon_t \le a_{t-1} - 1 - m_t < a_{t-1}$, according to Lemma 3, we have that $a_{t-1} + a_t + \cdots + a_{k-1} - 1 - m_t - m_{t+1} - \cdots - m_k - \varepsilon_t \le -1$, i.e., $a_{t-1} + a_t + \cdots + a_{k-1} \le m_t + m_{t+1} + \cdots + m_k + \varepsilon_t$.

Now, summing up inequalities (15), we get

$$\|\alpha\| \le \|\mu\| + \sum_{t=2}^{k} \varepsilon_t = \|\mu\| + i - 1 = \|\mu^{i-1}\| = \|\alpha_k\| - n - 1 = \|\alpha\| - k - n - 1 < \|\alpha\|,$$

which is obviously a contradiction. □

The set $G^{\mathbb{C}}$ was defined as the set of polynomials $g_\mu$ such that $|\mu| \le n+1$. As the first consequence of Proposition 7, let us prove that for any $(k-1)$-tuple $\mu$ of nonnegative integers, the polynomial $g_\mu$ belongs to the ideal generated by $G^{\mathbb{C}}$.

**Corollary 8.** *If $\mu = (m_2, \dots, m_k)$ is a $(k-1)$-tuple of nonnegative integers, then $g_\mu \in (G^{\mathbb{C}})$.*

Proof. Let us define a well ordering $\lessdot$ on the set $\mathbb{N}_0^{k-1}$ and prove the corollary by induction on $\lessdot$. For $(k-1)$-tuples of nonnegative integers $\mu = (m_2, \dots, m_k)$ and $\nu = (n_2, \dots, n_k)$, we first compare $|\mu|$ and $|\nu|$, that is, $m_2 + m_3 + \cdots + m_k$ and $n_2 + n_3 + \cdots + n_k$, and if these are equal, then we compare $m_3 + \cdots + m_k$ and $n_3 + \cdots + n_k$, and so on. More precisely, if $|\mu|_{i\to} := m_i + \cdots + m_k$, $2 \le i \le k$, then

$$\nu \lessdot \mu \quad \text{if and only if} \quad |\nu|_{s\to} < |\mu|_{s\to}, \text{ where } s = \min\{i : |\mu|_{i\to} \ne |\nu|_{i\to}\}.$$

If $|\mu| \le n+1$, then $g_\mu \in G^{\mathbb{C}} \subset (G^{\mathbb{C}})$. Suppose now that $|\mu| > n+1$ and that $g_\nu \in (G^{\mathbb{C}})$ for all $\nu$ such that $\nu \lessdot \mu$. Choose $i$ and $j$ such that $1 \le i \le j \le k-1$ and such that all components of $\mu_{i,j}$ are nonnegative (this is possible since $|\mu| > n+1$). By Proposition 7,

$$g_\mu = g_{\mu_{i,j}^{i,j}} = c_i g_{\mu_i} - c_{j+1} g_{\mu_{i,j}^{i-1}} + g_{\mu_{i,j}^{i-1,j+1}} \in (G^{\mathbb{C}}),$$

since obviously $\mu_{i,j}^{i-1} \lessdot \mu_i \lessdot \mu$, and (if $j \le k-2$) $\mu_{i,j}^{i-1,j+1} \lessdot \mu$. □

In the following lemma we establish a connection between polynomials $g_\mu$ and polynomials (dual classes) $\overline{c}_r \in \mathbb{Z}[c_1, c_2, \dots, c_k]$ from the previous section.

**Lemma 9.** *For $m \ge 0$ and $(k-1)$-tuple $\mathbf{m} = (m, 0, \dots, 0)$ we have that*

$$g_{\mathbf{m}} = (-1)^{n+1} \sum_{i=0}^{m} \binom{m}{i} c_1^{m-i} \cdot \overline{c}_{n+1+i}.$$

Proof. The polynomials $g_\mu$ were introduced in Definition 2 and they depend on the (previously fixed) integer $n$. In this proof (and only in this proof) we allow $n$ to vary through the set $\mathbb{N}$, while the integer $k$ is still fixed (we are working in the polynomial algebra $\mathbb{Z}[c_1, c_2, \dots, c_k]$). Note that the polynomials $\overline{c}_r$, $r \ge 1$, are defined independently of $n$. We emphasize the dependence of $g_\mu$ on $n$ by using an appropriate superscript, and we actually prove the following claim:

$$g_{\mathbf{m}}^{(n)} = (-1)^{n+1} \sum_{i=0}^{m} \binom{m}{i} c_1^{m-i} \cdot \overline{c}_{n+1+i}, \qquad \text{for all } m \ge 0 \text{ and all } n \ge 1.$$

The proof is by induction on $m$. We have already noticed that $g_{\mathbf{0}}^{(n)} = (-1)^{n+1} \overline{c}_{n+1}$, and therefore, the claim is true for $m = 0$ (and all $n \ge 1$). So, let $m \ge 1$ and assume that the claim is true for the integer $m-1$ and all $n \ge 1$. Let $\mathbf{m} = (m, 0, \dots, 0)$ and $n \in \mathbb{N}$. Note that $\mathbf{m}_1 = (m-1, 0, \dots, 0) = \mathbf{m} - \mathbf{1}$. Since, for all $k$-tuples $\alpha$ of integers, $[\alpha, \mathbf{m}]_2 = -[\alpha_1, \mathbf{m} - \mathbf{1}]_2 + [\alpha, \mathbf{m} - \mathbf{1}]_2$ by (5) and $[\alpha, \mathbf{m}]_t = [\alpha_1, \mathbf{m} - \mathbf{1}]_t = [\alpha, \mathbf{m} - \mathbf{1}]_t$

for $3 \le t \le k$, we have that

$$\begin{aligned}
g_{\mathbf{m}}^{(n)} &= \sum_{\|\alpha\|=n+1+m} (-1)^{n+1+|\alpha|}[\alpha,\mathbf{m}]C^{\alpha} \\
&= \sum_{\|\alpha\|=n+1+m} (-1)^{n+1+|\alpha|}\left(\Big(-[\alpha_1,\mathbf{m}-\mathbf{1}]_2+[\alpha,\mathbf{m}-\mathbf{1}]_2\Big)\prod_{t=3}^{k}[\alpha,\mathbf{m}]_t\right)C^{\alpha} \\
&= c_1 \sum_{\|\alpha_1\|=n+1+m-1} (-1)^{n+1+|\alpha_1|}[\alpha_1,\mathbf{m}-\mathbf{1}]C^{\alpha_1} - \sum_{\|\alpha\|=(n+1)+1+m-1} (-1)^{n+2+|\alpha|}[\alpha,\mathbf{m}-\mathbf{1}]C^{\alpha} \\
&= c_1 g_{\mathbf{m}-\mathbf{1}}^{(n)} - g_{\mathbf{m}-\mathbf{1}}^{(n+1)} \\
&= c_1\cdot(-1)^{n+1}\sum_{i=0}^{m-1}\binom{m-1}{i}c_1^{m-1-i}\cdot\overline{c}_{n+1+i} - (-1)^{n+2}\sum_{i=0}^{m-1}\binom{m-1}{i}c_1^{m-1-i}\cdot\overline{c}_{n+2+i} \\
&= (-1)^{n+1}\sum_{i=0}^{m}\binom{m-1}{i}c_1^{m-i}\cdot\overline{c}_{n+1+i} + (-1)^{n+1}\sum_{i=0}^{m}\binom{m-1}{i-1}c_1^{m-i}\cdot\overline{c}_{n+1+i} \\
&= (-1)^{n+1}\sum_{i=0}^{m}\binom{m}{i}c_1^{m-i}\cdot\overline{c}_{n+1+i},
\end{aligned}$$

and the proof is completed. □

**Proposition 10.** $I_{k,n} = (G^{\mathbb{C}})$.

Proof. Let us first prove that $(G^{\mathbb{C}}) \subseteq I_{k,n}$. Since the ideal $I_{k,n}$ is generated by the polynomials $\overline{c}_{n+1}, \overline{c}_{n+2}, \ldots, \overline{c}_{n+k}$, note that, by the recurrence relation (2), not only these $k$ polynomials, but all $\overline{c}_r$ for $r \ge n+1$ belong to $I_{k,n}$. Likewise, we shall prove that $g_\mu \in I_{k,n}$ for all $(k-1)$-tuples $\mu$ of nonnegative integers, and not only for those with the property $|\mu| \le n+1$ (i.e., $g_\mu \in G^{\mathbb{C}}$).

We define the relation $<_{lexr}$ on the set of all $(k-1)$-tuples of nonnegative integers by

$$(n_2, n_3, \ldots, n_k) <_{lexr} (m_2, m_3, \ldots, m_k) \Longleftrightarrow n_t < m_t, \text{ where } t = \max\{i \mid n_i \ne m_i\},$$

which is exactly the strict part of the lexicographical right ordering. This is a well ordering and our proof is by induction on $<_{lexr}$.

For the $(k-1)$-tuple $\mathbf{m} = (m, 0, \ldots, 0)$, where $m \ge 0$ is arbitrary integer, from Lemma 9 and our remark at the beginning of this proof, we immediately get that $g_{\mathbf{m}} \in I_{k,n}$. So, let us now take a $(k-1)$-tuple $\mu = (m_2, m_3, \ldots, m_k)$ such that the greatest integer $s$ with the property $m_{s+1} > 0$ is at least 2. Hence, $2 \le s \le k-1$ and $\mu = (m_2, \ldots, m_{s+1}, 0, \ldots, 0)$. Let us also assume that $g_\nu \in I_{k,n}$ for all $\nu$ such that $\nu <_{lexr} \mu$. We wish to prove that $g_\mu \in I_{k,n}$. By Proposition 7, applied to the $(k-1)$-tuple $\mu_s$, $i = 1$ and $j = s-1$,

$$g_\mu = g_{\mu_s^{1,s-1}} - c_1 g_{\mu_s^{s-1}} + c_s g_{\mu_s}.$$

Since $\mu_s <_{lexr} \mu_s^{s-1} <_{lexr} \mu_s^{1,s-1} <_{lexr} \mu$, we conclude that $g_\mu \in I_{k,n}$.

For the opposite inclusion ($I_{k,n} = (\overline{c}_{n+1}, \ldots, \overline{c}_{n+k}) \subseteq (G^{\mathbb{C}})$), we know that $\overline{c}_{n+1} = (-1)^{n+1} g_{\mathbf{0}} \in (G^{\mathbb{C}})$, and note that, by Lemma 9, $\overline{c}_{n+1+m} = (-1)^{n+1} g_{\mathbf{m}} - \sum_{i=0}^{m-1}\binom{m}{i}c_1^{m-i}\cdot\overline{c}_{n+1+i}$, $m \ge 1$. The statement now follows by induction. □

We are now ready to prove the main theorem of this section.

**Theorem 11.** *The set $G^{\mathbb{C}}$ is a minimal strong Gröbner basis for the ideal $I_{k,n}$ in $\mathbb{Z}[c_1, \ldots, c_k]$ with respect to grlex ordering $\preccurlyeq$.*

Proof. By Proposition 10, $G^{\mathbb{C}}$ is a basis for $I_{k,n}$. By Proposition 5, $0 \notin G^{\mathbb{C}}$ and it is obvious from the definition that $G^{\mathbb{C}}$ is finite. Suppose, to the contrary, that $G^{\mathbb{C}}$ is not a strong Gröbner basis for $I_{k,n}$. Then there is a polynomial $f \in I_{k,n} \setminus \{0\}$ such that $\mathrm{LT}(g) \nmid \mathrm{LT}(f)$ for all $g \in G^{\mathbb{C}}$. However, according to Proposition 5 again, the set $\{\mathrm{LT}(g) : g \in G^{\mathbb{C}}\}$ is exactly the set of all monomials in $\mathbb{Z}[c_1, c_2, \ldots, c_k]$ with the sum of the exponents equal to $n+1$, that is $\{\mathrm{LT}(g) : g \in G^{\mathbb{C}}\} = \{C^\lambda : |\lambda| = n+1\}$, which means that $|\alpha| \leq n$ for all $C^\alpha \in M(f)$. This is not possible, since, by Proposition 2, the set $B_{k,n}$ of all (cosets of) monomials $C^\alpha$ with $|\alpha| \leq n$ is an additive basis for the quotient algebra $\mathbb{Z}[c_1, \ldots, c_k]/I_{k,n} = H^*(G_{k,n}(\mathbb{C}); \mathbb{Z})$. So, $G^{\mathbb{C}}$ is a strong Gröbner basis for $I_{k,n}$.

By Proposition 5, distinct polynomials from $G^{\mathbb{C}}$ have distinct leading terms. Since $\{\mathrm{LT}(g) : g \in G^{\mathbb{C}}\} = \{C^\lambda : |\lambda| = n+1\}$, it is clear that no leading term in $G^{\mathbb{C}}$ divides some other leading term in $G^{\mathbb{C}}$, i.e., the strong Gröbner basis $G^{\mathbb{C}}$ is minimal. □

Propositions 5 and 7 enable us to explicitly determine the polynomials $g_\mu \in G^{\mathbb{C}}$ for the $(k-1)$-tuples $\mu = (m_2, \ldots, m_k)$ such that $m_k$ is close to $n$. Namely, if $g_\mu \in G^{\mathbb{C}}$ and $C^\alpha \in M(g_\mu) \setminus \{C^{\overline{\mu}}\}$ (where $\overline{\mu} = (n+1-|\mu|, m_2, \ldots, m_k)$), then $|\alpha| \leq n$ by Proposition 5. Consequently, $\|\alpha\| = \sum_{j=1}^k j a_j \leq k \sum_{j=1}^k a_j = k|\alpha| \leq kn$. On the other hand, $\|\alpha\| = n + 1 + \|\mu\|$, and so, we conclude that $g_\mu = C^{\overline{\mu}}$ whenever $\|\mu\| > (k-1)n - 1$.

Let $\nu$ be the $(k-1)$-tuple $(0, \ldots, 0, n)$. Since $\|\nu^s\| > \|\nu\| = (k-1)n$ $(1 \leq s \leq k-1)$, by the previous remark we have that

$$g_\nu = c_1 c_k^n \quad \text{and} \quad g_{\nu^s} = c_{s+1} c_k^n, \quad 1 \leq s \leq k-1 \quad (\text{for } k \geq 2). \tag{16}$$

If we apply Proposition 7 to the $(k-1)$-tuple $\nu_{k-1} = (0, \ldots, 0, n-1)$, $i = 1$ and $j = k-1$, we obtain the relation $c_k g_{\nu_{k-1}} = c_1 g_\nu - g_{\nu^1}$. Both summands on the right-hand side contain $c_k$ as a factor, so $c_k$ cancels out and using (16) we get

$$g_{\nu_{k-1}} = c_1^2 c_k^{n-1} - c_2 c_k^{n-1} \qquad (\text{for } k \geq 2). \tag{17}$$

Likewise, by applying Proposition 7 to $\nu_{k-1}$, $i = s+1$ and $j = k-1$, one obtains that $c_k g_{\nu^s_{k-1}} = c_{s+1} g_\nu - g_{\nu^{s+1}}$, and so

$$g_{\nu^s_{k-1}} = c_1 c_{s+1} c_k^{n-1} - c_{s+2} c_k^{n-1}, \quad 1 \leq s \leq k-2 \quad (\text{for } k \geq 3). \tag{18}$$

Identities (17) and (18) determine $g_\mu \in G^{\mathbb{C}}$ when $m_k = n-1$ and $|\mu| \leq n$. For computing $g_\mu \in G^{\mathbb{C}}$ when $m_k = n-1$ and $|\mu| = n+1$ for a concrete integer $k$, one can use Proposition 7 and apply it first to $\nu_{k-1}$, $i = 1$ and all $j$ such that $1 \leq j \leq k-2$, then to $\nu_{k-1}$, $i = 2$ and all $j$ such that $2 \leq j \leq k-2$ and so on. After that, if $n \geq 2$, the polynomials $g_\mu \in G^{\mathbb{C}}$ for $m_k = n-2$ can be obtained in the same manner – by suitable applications of Proposition 7. For example,

$$\begin{aligned} g_{\nu_{k-1,k-1}} &= c_1^3 c_k^{n-2} - 2c_1 c_2 c_k^{n-2} + c_3 c_k^{n-2} \qquad (\text{for } k \geq 3), \\ g_{\nu^s_{k-1,k-1}} &= c_1^2 c_{s+1} c_k^{n-2} - c_1 c_{s+2} c_k^{n-2} - c_2 c_{s+1} c_k^{n-2} + c_{s+3} c_k^{n-2}, \quad 1 \leq s \leq k-3 \quad (\text{for } k \geq 4). \end{aligned}$$

## 4. Pieri-type formula for $B_{k,n}$

Let $\lambda = (l_1, l_2, \ldots, l_k)$ be a $k$-tuple of nonnegative integers such that $|\lambda| = n+1$. Then, for $\underline{\lambda} := (l_2, \ldots, l_k)$ we have $g_{\underline{\lambda}} \in G^{\mathbb{C}}$, and so $g_{\underline{\lambda}} = 0$ in $H^*(G_{k,n}(\mathbb{C}); \mathbb{Z})$. Therefore

$$C^\lambda = \sum_{\substack{\|\alpha\| = n+1+\|\underline{\lambda}\| \\ \alpha \neq \lambda}} (-1)^{1+|\alpha|-|\lambda|} [\alpha, \underline{\lambda}]\, C^\alpha \tag{19}$$

in $H^*(G_{k,n}(\mathbb{C}); \mathbb{Z})$. By Proposition 5, this identity is in fact the presentation of $C^\lambda$ in the additive basis $B_{k,n}$ (see Section 2). Also, note that these formulas completely determine the multiplication in $H^*(G_{k,n}(\mathbb{C}); \mathbb{Z})$. Therefore, formula (19) can be understood as a Pieri-type formula for the elements of the basis $B_{k,n}$. More precisely, for $1 \leq i \leq k$ and $C^\lambda \in B_{k,n}$, if $|\lambda| < n$, then $c_i \cdot C^\lambda \in B_{k,n}$, and if $|\lambda| = n$, then

$$c_i \cdot C^\lambda = C^{\lambda^i} = \sum_{\substack{\|\alpha\| = n+1+\|\underline{\lambda^i}\| \\ \alpha \neq \lambda^i}} (-1)^{|\alpha|-|\lambda|} [\alpha, \underline{\lambda^i}]\, C^\alpha \tag{20}$$

For example, by the calculation at the end of the previous section, we have the following identities:

$$
\begin{aligned}
&c_1 c_k^n = 0,\\
&c_{s+1} c_k^n = 0, \quad 1 \le s \le k-1 \quad (k \ge 2),\\
&c_1^2 c_k^{n-1} = c_2 c_k^{n-1} \quad (k \ge 2),\\
&c_1 c_{s+1} c_k^{n-1} = c_{s+2} c_k^{n-1}, \quad 1 \le s \le k-2 \quad (k \ge 3),\\
&c_1^3 c_k^{n-2} = 2 c_1 c_2 c_k^{n-2} - c_3 c_k^{n-2} \quad (k \ge 3,\ n \ge 2),\\
&c_1^2 c_{s+1} c_k^{n-2} = c_1 c_{s+2} c_k^{n-2} + c_2 c_{s+1} c_k^{n-2} - c_{s+3} c_k^{n-2}, \quad 1 \le s \le k-3 \quad (k \ge 4,\ n \ge 2).
\end{aligned}
$$

## 5. Recurrence formulas for Kostka numbers

For a $k$-tuple $\alpha = (a_1, \ldots, a_k)$ of nonnegative integers, let $\alpha_{\rightarrow}$ denote the partition which has exactly $a_i$ components equal to $i$, for $1 \le i \le k$ (for example, if $\alpha = (3,2,0,3)$, then $\alpha_{\rightarrow} = (4,4,4,2,2,1,1,1)$). Note that $|\alpha| = l(\alpha_{\rightarrow})$ and $\|\alpha\| = |\alpha_{\rightarrow}|$.

By the identity (4), in $H^*(G_{k,n}(\mathbb{C}); \mathbb{Z})$ we have

$$
C^\alpha = (-1)^{\|\alpha\|} \sum_{\substack{|\lambda| = \|\alpha\| \\ \lambda \subset n \times k}} K_{\lambda \alpha_{\rightarrow}} \, \sigma_{\lambda^*},
$$

where the condition $\lambda \subset n \times k$ may be omitted, if we introduce the convention that $\sigma_{\lambda^*} = 0$ if $\lambda \not\subset n \times k$. Plugging these in the expression for $g_\mu$, where $\mu$ is a fixed $(k-1)$-tuple of nonnegative integers, and using the fact that $g_\mu = 0$ in $H^*(G_{k,n}(\mathbb{C}); \mathbb{Z})$, we obtain

$$
\begin{aligned}
0 &= \sum_{\|\alpha\| = n+1+\|\mu\|} (-1)^{n+1+|\alpha|} [\alpha, \mu] C^\alpha \\
&= \sum_{\|\alpha\| = n+1+\|\mu\|} (-1)^{n+1+|\alpha|+\|\alpha\|} [\alpha, \mu] \sum_{|\lambda| = \|\alpha\|} K_{\lambda \alpha_{\rightarrow}} \, \sigma_{\lambda^*} \\
&= \sum_{\|\alpha\| = n+1+\|\mu\|} \sum_{|\lambda| = \|\alpha\|} (-1)^{|\alpha| + \|\mu\|} [\alpha, \mu] K_{\lambda \alpha_{\rightarrow}} \, \sigma_{\lambda^*} \\
&= (-1)^{\|\mu\|} \sum_{|\lambda| = n+1+\|\mu\|} \left( \sum_{\|\alpha\| = |\lambda|} (-1)^{|\alpha|} [\alpha, \mu] K_{\lambda \alpha_{\rightarrow}} \right) \sigma_{\lambda^*}
\end{aligned}
$$

(in these sums $\alpha$'s are $k$-tuples of nonnegative integers and $\lambda$'s are partitions). Therefore, in view of the additive basis $\Sigma_{k,n}$, for every partition $\lambda \subset n \times k$ such that $|\lambda| = n + 1 + \|\mu\|$ we have

$$
\sum_{\|\alpha\| = |\lambda|} (-1)^{|\alpha|} [\alpha, \mu] K_{\lambda \alpha_{\rightarrow}} = 0. \tag{21}
$$

For the proof of the main result of this section we will need some additional notations. For a given $k \in \mathbb{N}$, let $\nu = (n_1, n_2, \ldots)$ be a partition such that $n_1 \le k$. Define $m_i := |\{j : n_j = i\}|$, $1 \le i \le k$. Then $\overline{\mu} := (m_1, m_2, \ldots, m_k)$ is the unique $k$-tuple of nonnegative integers such that $\nu = \overline{\mu}_{\rightarrow}$. Also, the $(k-1)$-tuple $\mu := (m_2, \ldots, m_k)$ satisfies

$$
\|\mu\| = \|\overline{\mu}\| - |\overline{\mu}| = |\overline{\mu}_{\rightarrow}| - l(\overline{\mu}_{\rightarrow}) = |\nu| - l(\nu).
$$

Note that, if $n := l(\nu) - 1$, then $m_1 = |\overline{\mu}| - |\mu| = l(\nu) - |\mu| = n + 1 - |\mu|$, so $\overline{\mu} = (n + 1 - |\mu|, m_2, \ldots, m_k)$ which, in view of Proposition 5, explains the notation $\overline{\mu}$ for this $k$-tuple. We now define

$$
\nu_{\leftarrow} = \nu_{\leftarrow}^{(k)} := \mu.
$$

**Theorem 12.** *Let $\lambda = (l_1, l_2, \ldots)$ and $\nu = (n_1, n_2, \ldots)$ be partitions such that $|\lambda| = |\nu|$. We have the following relations:*

*(i) if $\lambda = \nu = \emptyset$, then $K_{\lambda\nu} = 1$;*

*(ii) if $l(\lambda) < l(\nu)$ and $l_1 \geq n_1$, then*

$$K_{\lambda\nu} = \sum_{\substack{\|\alpha\|=|\lambda| \\ \alpha_\rightarrow \neq \nu}} (-1)^{1+l(\nu)+|\alpha|} [\alpha, \nu_\leftarrow] K_{\lambda\alpha_\rightarrow},$$

*where $\nu_\leftarrow = \nu_\leftarrow^{(l_1)}$, and the sum is over $l_1$-tuples $\alpha$ of nonnegative integers with the specified properties.*

*(iii) if $l(\lambda) = l(\nu) = s > 0$, then $K_{\lambda\nu} = K_{(l_1-1,\ldots,l_s-1)\,(n_1-1,\ldots,n_s-1)}$;*

*(iv) if $l(\lambda) > l(\nu)$ or $l_1 < n_1$, then $K_{\lambda\nu} = 0$.*

Proof. Note that, if $l(\lambda) > l(\nu)$ or $l_1 < n_1$, then $\lambda \not\geq \nu$, and therefore $K_{\lambda\nu} = 0$, which completes the proof of (iv).

Let $l(\lambda) = l(\nu) = s > 0$, and let us observe a semistandard Young tableau of shape $\lambda$ and type $\nu$. The first column of this tableau contains $s$ different numbers, and since $l(\nu) = s$, it can be filled in a unique way. Moreover, by removing the first column from this tableau, we obtain a Young tableau of size $(l_1-1, \ldots, l_s-1)$ and type $(n_1 - 1, \ldots, n_s - 1)$, which completes the proof of (iii).

So, let $l(\lambda) < l(\nu)$ and $l_1 \geq n_1$. Let $k := l_1$ and $n := l(\nu) - 1$. Then $\lambda \subset n \times k$ (since $l_1 = k$ and $l(\lambda) \leq l(\nu) - 1 = n$) and $|\lambda| = |\nu| = l(\nu) + \|\nu_\leftarrow\| = n + 1 + \|\nu_\leftarrow\|$. Therefore, by identity (21), we have

$$\sum_{\|\alpha\|=|\lambda|} (-1)^{|\alpha|} [\alpha, \nu_\leftarrow] K_{\lambda\alpha_\rightarrow} = 0. \tag{22}$$

If $\mu = \nu_\leftarrow = (m_2, \ldots, m_k)$, then for the $k$-tuple $\overline{\mu} = (n + 1 - |\mu|, m_2, \ldots, m_k)$ we have that $\overline{\mu}_\rightarrow = \nu$ and $\|\overline{\mu}\| = |\overline{\mu}_\rightarrow| = |\nu| = |\lambda|$. So,

$$(-1)^{|\overline{\mu}|} [\overline{\mu}, \nu_\leftarrow] K_{\lambda\nu}$$

is a summand of the previous sum. Since $|\overline{\mu}| = l(\nu)$, and, by Proposition 5, $[\overline{\mu}, \nu_\leftarrow] = [\overline{\mu}, \mu] = 1$, the identity in (ii) is obtained from (22). □

**Remark 4.** In the equation in part (ii) of the theorem, if a coefficient $[\alpha, \nu_\leftarrow]$ on the right-hand side is nonzero, then $\alpha_\rightarrow \neq \nu$, that is $\alpha \neq \overline{\mu} = \overline{\nu_\leftarrow}$, and $\|\alpha\| = |\lambda| = n + 1 + \|\nu_\leftarrow\|$ (see the proof of the theorem). By Proposition 5, we conclude that $l(\alpha_\rightarrow) = |\alpha| < n + 1 = l(\nu)$.

**Remark 5.** Theorem 12 gives us a recurrence relation which can be used to calculate (all) Kostka numbers. To prove this claim, let us define relation $<_s$ on the set of all partitions in the following way:

$$\nu' <_s \nu'' \quad \overset{def}{\Longleftrightarrow} \quad l(\nu') < l(\nu'') \quad \text{or else} \quad l(\nu') = l(\nu'') \text{ and } |\nu'| < |\nu''|.$$

Note that this relation is transitive, and that for any partition $\nu$ there does not exist infinite sequence $\{\nu_{(k)}\}_{k\in\mathbb{N}}$ of partitions that satisfies $\nu >_s \nu_{(1)} >_s \cdots >_s \nu_{(k)} >_s \cdots$.

Now, in case (ii) of Theorem 12, $K_{\lambda\nu}$ is, by the previous remark, a linear combination of elements $K_{\lambda\nu'}$ such that $\nu' <_s \nu$, and in case (iii) of Theorem 12 clearly $(n_1-1, \ldots, n_s-1) <_s \nu$. Therefore, an element $K_{\lambda\nu}$ is equal to zero or one, or can be expressed as a function of elements $K_{\lambda'\nu'}$ with $\nu' <_s \nu$, which completes the proof of our claim.

## 6. Gröbner bases for real Grassmannians

Borel's description of the mod 2 cohomology algebra of $G_{k,n}(\mathbb{C})$ is obtained from the corresponding description of the integral cohomology algebra by reducing modulo 2. Since the mod 2 reduction of the Chern class $c_i \in H^{2i}(G_{k,n}(\mathbb{C});\mathbb{Z})$ of the canonical (complex) bundle $\gamma_k$ over $G_{k,n}(\mathbb{C})$ is the Stiefel-Whitney class $w_{2i} \in H^{2i}(G_{k,n}(\mathbb{C});\mathbb{Z}_2)$ of the underlying real vector bundle, we have that the set $\{w_2^{a_1}w_4^{a_2}\cdots w_{2k}^{a_k} : a_1+a_2+\cdots+a_k \leq n\}$ is an additive basis for $H^*(G_{k,n}(\mathbb{C});\mathbb{Z}_2)$, and a Gröbner basis for the ideal in $\mathbb{Z}_2[w_2,w_4,\ldots,w_{2k}]$ which determines $H^*(G_{k,n}(\mathbb{C});\mathbb{Z}_2)$ can be obtained in the same way as in Section 3 by substituting $w_{2i}$ for $c_i$.

Nonetheless, the description of the mod 2 cohomology algebra of real Grassmann manifold $G_{k,n}(\mathbb{R})$ is essentially the same as the one for $G_{k,n}(\mathbb{C})$ – the only difference being in the fact that dimensions of the generating Stiefel-Whitney classes are divided by 2. Therefore, the same conclusions, concerning additive basis for $H^*(G_{k,n}(\mathbb{R});\mathbb{Z}_2)$ and Gröbner basis for the corresponding ideal, hold in this case. Let us discuss this briefly.

Let $\gamma_k$ be the canonical vector bundle over $G_{k,n}(\mathbb{R})$ and $w_1,w_2,\ldots,w_k$ its Stiefel-Whitney classes. The cohomology algebra $H^*(G_{k,n}(\mathbb{R});\mathbb{Z}_2)$ is isomorphic to the polynomial algebra $\mathbb{Z}_2[w_1,w_2,\ldots,w_k]$ modulo the ideal $J_{k,n}$ generated by the dual classes $\overline{w}_{n+1},\overline{w}_{n+2},\ldots,\overline{w}_{n+k}$. The analog of the formula (3) is the following explicit formula for these dual classes:

$$\overline{w}_r = \sum_{a_1+2a_2+\cdots+ka_k=r} [a_1,a_2,\ldots,a_k]w_1^{a_1}w_2^{a_2}\cdots w_k^{a_k}, \quad r \geq 1 \tag{23}$$

(since we are now working with $\mathbb{Z}_2$ coefficients, we are ignoring the signs and multinomial coefficients are considered mod 2).

For a $k$-tuple $\alpha = (a_1,a_2,\ldots,a_k)$ of nonnegative integers, let $W^\alpha := w_1^{a_1}w_2^{a_2}\cdots w_k^{a_k}$. By the previous discussion, first we have the following proposition.

**Proposition 13.** *The set $D_{k,n} = \{W^\alpha : |\alpha| \leq n\}$ is a vector space basis for*

$$H^*(G_{k,n}(\mathbb{R});\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1,w_2,\ldots,w_k]/J_{k,n}.$$

**Definition 3.** For a $(k-1)$-tuple $\mu = (m_2,\ldots,m_k)$ of nonnegative integers let

$$\widetilde{g}_\mu := \sum_{\|\alpha\|=n+1+\|\mu\|} [\alpha,\mu]\ W^\alpha,$$

where the sum is taken over all $k$-tuples of nonnegative integers $\alpha = (a_1,a_2,\ldots,a_k)$ such that $\|\alpha\| = n+1+\|\mu\|$.

Morover, let

$$G^{\mathbb{R}} := \{\widetilde{g}_\mu : |\mu| \leq n+1\}.$$

So, we know that $G^{\mathbb{R}}$ is a minimal (strong) Gröbner basis for the ideal $J_{k,n}$ in $\mathbb{Z}_2[w_1,\ldots,w_k]$ with respect to the grlex ordering on monomials with $w_1 > w_2 > \cdots > w_k$. Furthermore, this Gröbner basis is not just minimal, it is reduced. Since $\mathrm{LC}(\widetilde{g}_\mu) = 1$ for all $\widetilde{g}_\mu \in G^{\mathbb{R}}$, to prove this claim it is enough to prove that for a given $\widetilde{g}_\mu \in G^{\mathbb{R}}$, $\mathrm{LT}(\widetilde{g}_\mu) = W^{\overline{\mu}}$ does not divide any monomial from $M(\widetilde{g}_\nu)$, for any $\widetilde{g}_\nu \in G^{\mathbb{R}} \setminus \{\widetilde{g}_\mu\}$. This is evident from Proposition 5 (that is, its mod 2 variant). Namely, we know that $W^{\overline{\mu}} \nmid W^{\overline{\nu}}$ ($G^{\mathbb{R}}$ is minimal) and if $W^\alpha$ is some other monomial in $\widetilde{g}_\nu$, then $|\alpha| < n+1$, so $W^\alpha$ cannot be divisible by $W^{\overline{\mu}}$ since $|\overline{\mu}| = n+1$.

We summarize this discussion in the following main theorem of this section.

**Theorem 14.** *The set $G^{\mathbb{R}}$ is the reduced Gröbner basis for the ideal $J_{k,n}$ in $\mathbb{Z}_2[w_1,w_2,\ldots,w_k]$ with respect to the grlex ordering on monomials with $w_1 > w_2 > \cdots > w_k$.*

As in Section 4, we can completely determine the multiplication in $H^*(G_{k,n}(\mathbb{R});\mathbb{Z}_2)$ by a Pieri-type formula for the elements of the additive basis $D_{k,n}$. If $\lambda = (l_1, l_2, \ldots, l_k)$ is a $k$-tuple of nonnegative integers such that $|\lambda| = n+1$, then

$$W^\lambda = \sum_{\substack{\|\alpha\|=n+1+\|\underline{\lambda}\| \\ \alpha\neq\lambda}} [\alpha, \underline{\lambda}]\ W^\alpha, \tag{24}$$

where $\underline{\lambda} = (l_2, \ldots, l_k)$. So, the product of $w_i$ $(1 \le i \le k)$ with $W^\lambda \in D_{k,n}$, written as the linear combination of the basis elements from $D_{k,n}$, is as follows: if $|\lambda| < n$, then $w_i \cdot W^\lambda \in D_{k,n}$, and if $|\lambda| = n$, then

$$w_i \cdot W^\lambda = W^{\lambda^i} = \sum_{\substack{\|\alpha\|=n+1+\|\underline{\lambda}^i\| \\ \alpha\neq\lambda^i}} [\alpha, \underline{\lambda}^i]\ W^\alpha. \tag{25}$$

## 7. Application to immersions

In this section we consider the (real) Grassmannians $G_{5,n} = G_{5,n}(\mathbb{R})$, where $n$ is divisible by 8. As before, $w_i \in H^i(G_{5,n};\mathbb{Z}_2)$, $1 \le i \le 5$, is the $i$-th Stiefel-Whitney class of the canonical bundle $\gamma_5$ over $G_{5,n}$.

**Lemma 15.** *Let $n \equiv 0 \pmod 8$ and let $\nu$ be the stable normal bundle over Grassmann manifold $G_{5,n}$. Then for the Stiefel-Whitney classes of this bundle, the following equalities hold: $w_2(\nu) = w_1^2 + w_2$ and $w_i(\nu) = 0$ when $i \geqslant 5n-4$.*

Proof. Let $r \geqslant 3$ be the integer such that $2^r < n+5 \leqslant 2^{r+1}$. Note that this implies $n \geqslant 2^r$ since $n \equiv 0 \pmod 8$. In [7, p. 365] Hiller and Stong proved that

$$w(\nu) = w(\gamma_5 \otimes \gamma_5) \cdot (1 + w_1 + w_2 + w_3 + w_4 + w_5)^{2^{r+1}-n-5}, \tag{26}$$

and that the top nonzero class in this expression is in dimension $20 + 5(2^{r+1} - n - 5)$. Since $n \geqslant 2^r$, we have that $20 + 5(2^{r+1} - n - 5) \leqslant 20 + 5(2^r - 5) = 5 \cdot 2^r - 5 \leqslant 5n - 5$. This proves the second equality in the statement of the lemma.

For the first one, we need the fact $w_1(\gamma_5 \otimes \gamma_5) = w_2(\gamma_5 \otimes \gamma_5) = 0$, which is not hard to check by the method described in [10, Problem 7-C]. Using this fact and (26), one obtains that

$$w_2(\nu) = \binom{2^{r+1} - n - 5}{2} w_1^2 + (2^{r+1} - n - 5) w_2 = w_1^2 + w_2,$$

since $2^{r+1} - n - 5 \equiv 3 \pmod 8$. □

**Theorem 16.** *If $n \equiv 0 \pmod 8$, then $G_{5,n}$ immerses into $\mathbb{R}^{10n-3}$.*

Proof. Since $\dim G_{5,n} = 5n$, in order to prove that there is an immersion of $G_{5,n}$ into $\mathbb{R}^{10n-3}$, it suffices to show that the classifying map $f_\nu : G_{5,n} \to BO$ of the stable normal bundle $\nu$ over $G_{5,n}$ lifts up to $BO(5n-3)$ .

$$\begin{tikzcd} & BO(5n-3) \arrow[d, "p"] \\ G_{5,n} \arrow[ur, dashed] \arrow[r, "f_\nu"] & BO \end{tikzcd}$$

We shall lift the map $f_\nu$ by the standard procedure. We use the appropriate modified Postnikov tower for the fibration $p : BO(5n-3) \to BO$ (presented in the following diagram) and we need to lift the map $f_\nu$

up to level 3 of the tower. The table given below contains the relations that produce the $k$-invariants of the tower.

$$\begin{tikzcd}
 & BO(5n-3) \arrow[d] & & \\
 & E_2 \arrow[r, "k_1^2"] \arrow[d] & K(\mathbb{Z}_2, 5n) & \\
 & E_1 \arrow[r, "k_1^1 \times k_2^1"] \arrow[d] & K(\mathbb{Z}_2, 5n-1) \times K(\mathbb{Z}_2, 5n) & \\
G_{5,n} \arrow[r, "f_\nu"] \arrow[uuur, dashed] \arrow[uur, dashed] \arrow[ur, dashed] & BO \arrow[r, "w_{5n-2} \times w_{5n}"] & K(\mathbb{Z}_2, 5n-2) \times K(\mathbb{Z}_2, 5n) &
\end{tikzcd}$$

| |
|---|
| $k_1^1 : (Sq^2 + w_2)w_{5n-2} = 0$ |
| $k_2^1 : (Sq^2 + w_1^2 + w_2)Sq^1 w_{5n-2} + Sq^1 w_{5n} = 0$ |
| $k_1^2 : (Sq^2 + w_2)k_1^1 + Sq^1 k_2^1 = 0$ |

According to Lemma 15, $w_{5n-2}(\nu) = w_{5n}(\nu) = 0$, so, we can lift $f_\nu$ up to $E_1$. Observe now the relations in the table for $k_2^1$ and $k_1^2$. The fact

$$Sq^1(w_4 w_5^{n-1}) = (w_1 w_4 + w_5)w_5^{n-1} + w_4(n-1)w_1 w_5^{n-1} = n w_1 w_4 w_5^{n-1} + w_5^n = w_5^n \neq 0$$

in $H^{5n}(G_{5,n}; \mathbb{Z}_2) \cong \mathbb{Z}_2$ (Proposition 13) ensures that chosen liftings can always be perturbed so that the obstructions coming from these two $k$-invariants vanish. In order to overcome the obstruction induced by $k_1^1$, we study the map $\big(Sq^2 + w_2(\nu)\big) : H^{5n-3}(G_{5,n}; \mathbb{Z}_2) \to H^{5n-1}(G_{5,n}; \mathbb{Z}_2)$. Since $H^{5n-1}(G_{5,n}; \mathbb{Z}_2) \cong \mathbb{Z}_2$, it suffices to prove that this map is nontrivial. By formulas of Wu and Cartan, Lemma 15 and the fact that the polynomials from (**??**) are trivial in $H^*(G_{5,n}; \mathbb{Z}_2)$ (since they belong to Gröbner basis $G^{\mathbb{R}} \subset J_{5,n}$) we have

$$\begin{aligned}
\big(Sq^2 + w_2(\nu)\big)\,(w_2 w_5^{n-1}) &= (Sq^2 + w_1^2 + w_2)(w_2 w_5^{n-1}) \\
&= w_2^2 w_5^{n-1} + (w_1 w_2 + w_3)(n-1)w_1 w_5^{n-1} \\
&\quad + w_2\left((n-1)w_2 w_5^{n-1} + \binom{n-1}{2} w_1^2 w_5^{n-1}\right) + w_1^2 w_2 w_5^{n-1} + w_2^2 w_5^{n-1} \\
&= w_1^2 w_2 w_5^{n-1} + w_1 w_3 w_5^{n-1} + w_2^2 w_5^{n-1} \\
&= w_2 \widetilde{g}_{(0,0,0,n-1)} + \widetilde{g}_{(0,1,0,n-1)} + w_4 w_5^{n-1} \\
&= w_4 w_5^{n-1},
\end{aligned}$$

and this class is nonzero by Proposition 13. □

By the famous result of Cohen ([3]), Grassmannian $G_{5,n}$ can be immersed into $\mathbb{R}^{10n-\alpha(5n)}$, where $\alpha(5n)$ denotes the number of ones in the binary expansion of $5n$. This means that Theorem 16 improves this result whenever $\alpha(5n) = 2$ (and $n \equiv 0 \pmod 8$). Such a case occurs when $n$ is a power of two, and it is known that then $G_{5,n}$ cannot be immersed into $\mathbb{R}^{10n-6}$ ([7, p. 365]). So, if $n \geq 8$ is a power of two, then for $\mathrm{imm}(G_{5,n}) = \min\{d \mid G_{5,n} \text{ immerses into } \mathbb{R}^d\}$ the following inequalities hold

$$10n - 5 \leqslant \mathrm{imm}(G_{5,n}) \leqslant 10n - 3.$$

Actually, a sufficient and necessary condition for $\alpha(5n) = 2$ and $n \equiv 0 \pmod 8$ is that $n$ is of the form $2^r + \sum_{i=0}^{s}(2^{r+2+4i} + 2^{r+3+4i})$, $r \geqslant 3$, $s \geqslant -1$ (where the case $s = -1$ corresponds to the case $n = 2^r$).

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