Characteristic rank of canonical vector bundles over oriented Grassmann manifolds $\tilde{G}_{3,n}$

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Abstract

We determine the characteristic rank of the canonical oriented vector bundle over $\tilde{G}_{3,n}$ for all $n \geq 3$, and as a consequence, we obtain the affirmative answer to a conjecture of Korbaš and Rusin. As an application of this result, we calculate the $\mathbb{Z}_2$-cup-length for a new infinite family of manifolds $\tilde{G}_{3,n}$. This result confirms the corresponding claim of Fukaya’s conjecture.

1 Introduction

The characteristic rank of a real vector bundle $\alpha$ over a $d$-dimensional CW-complex $X$, denoted by charrank($\alpha$), is defined in [9] as the maximal integer $q \in \{0, 1, \ldots, d\}$ such that all cohomology classes in $H^j(X; \mathbb{Z}_2)$ for $0 \leq j \leq q$ are polynomials in Stiefel–Whitney classes $w_1(\alpha), w_2(\alpha), \ldots$ of the bundle $\alpha$.

There has been much work done recently in studying the characteristic rank of various vector bundles (see [5, 6, 9]), and especially the canonical vector bundle $\tilde{\gamma}_{k,n}$ over Grassmann manifold $\tilde{G}_{k,n}$ ($k \leq n$) of oriented $k$-dimensional subspaces in $\mathbb{R}^{n+k}$ (see [4, 7, 8, 12]). The majority of the obtained results pertains to the case $k = 3$. As the main result of this paper, we determine the exact value of charrank($\tilde{\gamma}_{3,n}$) for all $n \geq 3$. This is stated in the following theorem, which is proven in Section 3. (In the rest of the paper we assume that $n \geq 3$.)

Theorem 1.1 If $t \geq 3$ is the unique integer such that $2^{t-1} \leq n + 3 < 2^t$, then

$$\text{charrank}(\tilde{\gamma}_{3,n}) = \min \{3n - 2^t + 7, 2^t - 5\}.$$  

The technique of the proof combines the method of Korbaš and Rusin used in [7, 12] (for obtaining the lower bound for charrank($\tilde{\gamma}_{3,n}$)) with Gröbner bases for ”unoriented” Grassmann manifolds $G_{3,n}$ constructed in [11] (for obtaining the upper bound for charrank($\tilde{\gamma}_{3,n}$)).

An immediate corollary of Theorem 1.1 is the positive answer to Conjecture 3.3 from [7] (see Remark 3.5).

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The $\mathbb{Z}_2$-cup-length of a path connected space $X$, denoted by $\text{cup}(X)$, is defined as the maximal $r$ such that there exist classes $x_1, x_2, \ldots, x_r \in \tilde{H}^*(X; \mathbb{Z}_2)$ with nontrivial cup product $(x_1x_2 \cdots x_r \neq 0)$. In [2] Fukaya studied the $\mathbb{Z}_2$-cup-length of Grassmanians $G_{3,n}$ and conjectured [2, Conjecture 1.2] that for $n$ in approximately the first half of the interval $[2^{t-1} - 4, 2^t - 4)$ (where $t \geq 4$) this cup-length is equal to $2^{t-1} - 3$. In that paper he obtains Gröbner bases for certain ideals related to Grassmanians $\tilde{G}_{3,2^{t-1}-4}$ and uses them to verify this conjecture for $n = 2^{t-1} - 4$. Later, in [4, 7] Kobaš and Rusin verified it for $n \in \{2^{t-1} - 3, 2^{t-1} - 2, 2^{t-1} - 1, 2^{t-1}\}$ ($t \geq 5$ in the case $n = 2^{t-1}$). We prove that the conjecture is true for all $n$ in the first third or the interval $[2^{t-1} - 4, 2^t - 4)$. This result is stated in the following theorem, the proof of which occupies Section 4 of the paper.

**Theorem 1.2** Let $t \geq 3$ be the unique integer such that $2^{t-1} - 4 \leq n < 2^t - 4$. If $n \leq 2^{t-1} - 4 + \frac{2^{t-1}}{3}$, then:

(a) $\text{cup}(\tilde{G}_{3,n}) = 2^{t-1} - 3$;

(b) $\text{ht}(w_2(\tilde{\gamma}_{3,n})) = 2^{t-1} - 4$.

In the theorem, $\text{ht}(w_2(\tilde{\gamma}_{3,n}))$ is, as usual, the height of the class $w_2(\tilde{\gamma}_{3,n})$, that is, the maximal $m$ such that $w_2(\tilde{\gamma}_{3,n})^m \neq 0$.

## 2 Cohomology of Grassmann manifolds

In this paper, all cohomology groups are assumed to have coefficients in $\mathbb{Z}_2$.

Let $G_{3,n}$ be the Grassmann manifold of unoriented three-dimensional subspaces in $\mathbb{R}^{n+3}$. By the Borel description [1], the cohomology algebra $H^*(G_{3,n})$ is isomorphic to the quotient

$$\mathbb{Z}_2[w_1, w_2, w_3]/J_{3,n},$$

where $J_{3,n} = (\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+3})$ is the ideal generated by dual classes. Hence, these dual classes are polynomials in variables $w_1$, $w_2$ and $w_3$, and they are obtained from the relation

$$(1 + w_1 + w_2 + w_3)(1 + \varpi_1 + \varpi_2 + \cdots) = 1.$$

In the above isomorphism the classes of the variables $w_1$, $w_2$ and $w_3$ correspond to the Stiefel–Whitney classes of the canonical vector bundle $\gamma_{3,n}$ over $G_{3,n}$ (which, by an abuse of notation, we also denote by $w_1$, $w_2$ and $w_3$).

The proof of the following theorem can be found in [3].

**Theorem 2.1** The set $D_{3,n} = \{w_1^aw_2^bw_3^c : a + b + c \leq n\}$ is a vector space basis for $H^*(G_{3,n})$.

In [11] a Gröbner basis for the ideal $J_{3,n}$ is obtained. It consists of the polynomials $\tilde{g}_{m,l}$, indexed by the pairs of nonnegative integers $(m, l)$ such that $m + l \leq n + 1$. They are defined by

$$\tilde{g}_{m,l} := \sum_{a+2b+3c=n+1+m+2l} \binom{a + b + c - m - l}{a}
\binom{b + c - l}{b} w_1^a w_2^b w_3^c,$$

where the sum is taken over all triples $(a, b, c)$ of nonnegative integers such that $a + 2b + 3c = n + 1 + m + 2l$ ([11, p. 80]; see also [10, p. 3]). It is clear that the monomial $w_1^{n+1-m-l} w_2^m w_3^n$ appears
in $\tilde{g}_{m,l}$ with coefficient 1, and it is a fact ([11, Proposition 5]; see also [10, Proposition 2.5]) that all other monomials of $\tilde{g}_{m,l}$ have degree (sum of the exponents) at most $n$. Since $\tilde{g}_{m,l} = 0$ in $H^*(G_{3,n})$, this means that (1) gives us the representation of the monomial $w_1^{n+1-m-l}w_2^m w_3^l$ in the additive basis $D_{3,n}$:

$$w_1^{n+1-m-l}w_2^m w_3^l = \sum_{a+2b+3c=n+1+m+2l \atop (a,b,c) \neq (n+1-m-l,m,l)} \left( \frac{a+b+c-m-l}{a} \right) \left( \frac{b+c-l}{b} \right) w_1^a w_2^b w_3^c. \quad (2)$$

This is an equality in $H^{n+1+m+2l}(G_{3,n})$, and in the rest of the paper, we say that an element of $H^j(G_{3,n})$ (or $H^j(\tilde{G}_{3,n})$) has dimension $j$.

The obvious map $p : \tilde{G}_{3,n} \to G_{3,n}$ (which forgets the orientation of a three-dimensional subspace in $\mathbb{R}^{n+3}$) is a two-sheeted covering map, and it is well known that the associated Gysin exact sequence is of the form

$$\cdots \to H^j(G_{3,n}) \xrightarrow{p^*} H^j(\tilde{G}_{3,n}) \xrightarrow{H^j} H^{j+1}(G_{3,n}) \xrightarrow{p^*} \cdots, \quad (3)$$

where $H^j(G_{3,n}) \xrightarrow{w_1} H^{j+1}(G_{3,n})$, $j \geq 0$, is the homomorphism given with $\sigma \mapsto w_1{\sigma}$, $\sigma \in H^j(G_{3,n})$.

The canonical bundle $\gamma_{3,n}$ over $G_{3,n}$ pulls back via $p$ to the canonical bundle $\tilde{\gamma}_{3,n}$ over $\tilde{G}_{3,n}$, and therefore, $w_i(\tilde{\gamma}_{3,n}) = p^*w_i$, $i = 1, 2, 3$. Since every cohomology class in $H^*(G_{3,n})$ is a polynomial in $w_1, w_2$ and $w_3$, we have that charrank($\tilde{\gamma}_{3,n}$) $\geq q$ if and only if $p^*: H^j(G_{3,n}) \to H^j(\tilde{G}_{3,n})$ is onto for all $j \in \{0, 1, \ldots, q\}$ (where $0 \leq q \leq 3n = \dim \tilde{G}_{3,n}$). The following equivalence is now straightforward from the exactness of sequence (3):

$$\text{charrank}(\tilde{\gamma}_{3,n}) \geq q \iff H^j(G_{3,n}) \xrightarrow{w_1} H^{j+1}(G_{3,n}) \text{ is a monomorphism for all } j \in \{0, 1, \ldots, q\}. \quad (4)$$

Using this equivalence and Theorem 2.1, it is easy to see that

$$\text{charrank}(\tilde{\gamma}_{3,n}) \geq n - 1. \quad (5)$$

3 Proof of Theorem 1.1

We divide the proof of our main result into two parts. In the first we show that $\text{charrank}(\tilde{\gamma}_{3,n}) \leq \min(3n - 2^t + 7, 2^t - 5)$, and in the second that $\text{charrank}(\tilde{\gamma}_{3,n}) \geq \min(3n - 2^t + 7, 2^t - 5)$.

3.1 Upper bound

The following lemma is proved in [2, Proposition 3.2].

**Lemma 3.1** Let $t \geq 3$ be an integer. Then for all nonnegative integers $b$ and $c$ such that $2b + 3c = 2^{t-1} - 3$ the number $(\frac{b+c}{b})$ is divisible by 2.

**Proposition 3.2** If $t \geq 3$ is the unique integer such that $2^{t-1} \leq n + 3 < 2^t$, then

$$\text{charrank}(\tilde{\gamma}_{3,n}) \leq \min(3n - 2^t + 7, 2^t - 5).$$
PROOF — Suppose first that $3n - 2^t + 7 \leq 2^t - 5$, i.e., $n < 2^{t-1} - 4 + \frac{2^{t-1}}{3}$. We need to show that $\text{charrank}(\tilde{\gamma}_{3,n}) \leq 3n - 2^t + 7$.

Let $l = n - 2^{t-1} + 4$. Note that $1 \leq l \leq n$. Namely, since $n + 3 \geq 2^{t-1}$, it holds $l = n + 4 - 2^{t-1} \geq 1$; also, since $t \geq 3$, we have that $2^{t-1} \geq 4$, so $l = n + 4 - 2^{t-1} \leq n$. Equation (2) for this $l$ and $m = 0$ gives us that

$$w_1^{n+1-l}w_3 = \sum_{a+2b+3c=n+1+2l\atop (a,b,c)\neq(n+1-l,0,l)} \left(\frac{a+b+c-l}{a}\right)\left(\frac{b+c-l}{b}\right)w_1^aw_2^bw_3^c. \quad (6)$$

Note that if $w_2^bw_3^c$ is a monomial with nonzero coefficient in this sum (i.e., if the integer $\left(\frac{a+b+c-l}{a}\right)\left(\frac{b+c-l}{b}\right)$ is odd), then $c \geq l$ and $a \geq 1$.

Indeed, if $c < l$, then since $\left(\frac{b+c-l}{b}\right) \neq 0$, we have that $b + c - l < 0$, and now, since $\left(\frac{a+b+c-l}{a}\right) \neq 0$, it must be $a + b + c - l < 0$. But then $n + 1 + 2l = a + 2b + 3c < 3(a + b + c) < 3l$, which leads to the contradiction $l > n + 1$.

Assume now that $a = 0$. Since then $2b + 3(c-l) = n + 1 + 2l - 3l = n + 1 - l = 2^{t-1} - 3$, we can apply Lemma 3.1 to (nonnegative) integers $b$ and $c - l$. Thus we obtain that $\left(\frac{b+c-l}{b}\right)$ is even, which is a contradiction.

So, all monomials with nonzero coefficient in (6) are divisible by $w_1$. Hence, in $H^{n+1+2l}(G_{3,n})$, we have the equality

$$w_1\left(w_1^{n-l}w_3^l + \sum_{a+2b+3c=n+1+2l\atop (a,b,c)\neq(n+1-l,0,l)} \left(\frac{a+b+c-l}{a}\right)\left(\frac{b+c-l}{b}\right)w_1^aw_2^bw_3^c\right) = 0.$$ 

Since the expression in the brackets is a nontrivial linear combination of elements of the set $D_{3,n}$, by Theorem 2.1 it is a nonzero element in the kernel of $H^{n+1+2l}(G_{3,n}) \xrightarrow{w_1} H^{n+1+2l}(G_{3,n})$.

By (4),

$$\text{charrank}(\tilde{\gamma}_{3,n}) \leq n + 2l - 1 = 3n - 2^t + 7.$$ 

Suppose now that $3n - 2^t + 7 \geq 2^t - 5$, i.e., $n > 2^{t-1} - 4 + \frac{2^{t-1}}{3} (> 2^{t-1} - 3)$. We now want to prove that $\text{charrank}(\tilde{\gamma}_{3,n}) \leq 2^t - 5$.

Let $m = 2^t - 4 - n$. It is obvious that $m \geq 0$ (since $n \leq 2^t - 4$), and we also have that $m \leq n$ (since this is equivalent to $n \geq 2^{t-1} - 2$). Therefore, $0 \leq m \leq n$, and so, we can use equation (2) for this $m$ and $l = 0$:

$$w_1^{n+1-m}w_2^m = \sum_{a+2b+3c=n+1+m\atop (a,b,c)\neq(n+1-m,m,0)} \left(\frac{a+b+c-m}{a}\right)\left(\frac{b+c}{b}\right)w_1^aw_2^bw_3^c.$$ 

Since $n + 1 + m = 2^t - 3$, by Lemma 3.1, for every summand with $a = 0$ in this sum, the coefficient $\left(\frac{b+c}{b}\right)$ is even, and so, such a summand vanishes. Therefore, all monomials that appear in the sum are divisible by $w_1$. Similarly as in the first part of the proof, in $H^{n+1+m}(G_{3,n})$ we now have the relation

$$w_1\left(w_1^{n-m}w_2^m + \sum_{a+2b+3c=n+1+m\atop (a,b,c)\neq(n+1-m,m,0)} \left(\frac{a+b+c-m}{a}\right)\left(\frac{b+c}{b}\right)w_1^{n-m}w_2^b w_3^c\right) = 0,$$ 

which, by Theorem 2.1, leads to a nontrivial element in the kernel of the homomorphism $H^{n+m}(G_{3,n}) \xrightarrow{w_1} H^{n+1+m}(G_{3,n})$. By (4) this implies that $\text{charrank}(\tilde{\gamma}_{3,n}) \leq n + m - 1 = 2^t - 5$. \qed
3.2 Lower bound

Let us recall some notation from [4, 7, 8, 12]. For \( i \geq 1 \), let \( g_i \in \mathbb{Z}_2[w_2, w_3] \cong \mathbb{Z}_2[w_1, w_2, w_3]/(w_1) \) denote the reduction of the polynomial (dual class) \( \overline{w}_i \) modulo \( w_1 \). The corresponding polynomial in Stiefel–Whitney classes \( w_2 \) and \( w_3 \) in \( H^i(G_3, n) \) is again denoted by the same symbol. The main result of this subsection is based on [7, Proposition 2.4]. We are stating only a part of that proposition, and only for \( k = 3 \).

**Proposition 3.3** For an integer \( x \geq 0 \) observe the following set of polynomials in \( H^{n+1+x}(G_3, n) \):

\[
N_x(G_3, n) = \bigcup_{i=0}^{2} \{ w_2^b w_3^c g_{n+1+i} : 2b + 3c = x - i \}.
\]

If \( x \leq n - 1 \) and the set \( N_x(G_3, n) \) is linearly independent, then

\[
H^{n+x}(G_3, n) \xrightarrow{w_1} H^{n+1+x}(G_3, n)
\]

is a monomorphism.

In the polynomial algebra \( \mathbb{Z}_2[w_2, w_3] \), for all \( i \geq 4 \) one has the following recurrence relation (see [12, (2.3)])

\[
g_i = w_2 g_{i-2} + w_3 g_{i-3},
\]

which can also be written in the matrix form (see [12, p. 55]):

\[
\begin{pmatrix}
g_i \\
g_{i-1} \\
g_{i-2}
\end{pmatrix} = \begin{pmatrix}
  0 & w_2 & w_3 \\
  1 & 0 & 0 \\
  0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
g_{i-1} \\
g_{i-2} \\
g_{i-3}
\end{pmatrix}.
\]

This identity implies that for all integers \( r > 0 \) and \( i \geq r + 3 \), one has

\[
\begin{pmatrix}
g_i \\
g_{i-1} \\
g_{i-2}
\end{pmatrix} = \begin{pmatrix}
  0 & w_2 & w_3 \\
  1 & 0 & 0 \\
  0 & 1 & 0
\end{pmatrix}^r \begin{pmatrix}
g_{i-r} \\
g_{i-r-1} \\
g_{i-r-2}
\end{pmatrix}.
\]

(7)

In the remainder of this section we use the following notation:

\[
A = \begin{pmatrix}
  0 & w_2 & w_3 \\
  1 & 0 & 0 \\
  0 & 1 & 0
\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}
  0 & w_3 & 0 \\
  0 & 0 & w_1 \\
  1 & 0 & w_2
\end{pmatrix}.
\]

Note that \( AB = w_3 I = BA \), where \( I \) is the identity matrix, and hence \( A^r B^r = w_3^r I \), for all \( r > 0 \).

We will also need the following facts, which hold in \( \mathbb{Z}_2[w_2, w_3] \):

- \( g_i = 0 \) if and only if \( i = 2^{t-1} - 3 \) for some \( t \geq 3 \) ([4, Lemma 2.3(i)]);
- for all \( t \geq 3 \), \( g_{2^t-1-2} \) and \( g_{2^t-1-1} \) are coprime ([12, Lemma 2.5]).

**Proposition 3.4** If \( t \geq 3 \) is the unique integer such that \( 2^{t-1-3} < n + 3 < 2^t \), then

\[
\text{charrank}(\tilde{\gamma}_{3, n}) \geq \min\{3n - 2^t + 7, 2^t - 5\}.
\]

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PROOF — Let $\delta = \min\{3n - 2^t + 7, 2^t - 5\}$. By (5) and (4) it suffices to prove that

$$H^{n+\bar{x}}(G_{3,n}) \xrightarrow{w_1} H^{n+1+\bar{x}}(G_{3,n})$$

is a monomorphism for all $x \in \{0, 1, \ldots, \delta - n\}$. Note that $x \leq \delta - n$ implies $x \leq n - 1$. Indeed, if $n = 2^t-1 - 3$, then $x \leq \delta - n \leq 3n - 2^t + 7 - n = 1 < n - 1$; and if $n \geq 2^t-1 - 2$, then $x \leq \delta - n \leq 2^t - 5 - n \leq n - 1$. So, by Proposition 3.3, it is now enough to show that $N_x(G_{3,n})$ is linearly independent for all $x \in \{0, 1, \ldots, \delta - n\}$.

Let $0 \leq x \leq \delta - n$. If some linear combination of elements of the set $N_x(G_{3,n})$ vanishes, then in $H^{n+1+\bar{x}}(G_{3,n})$ one has the equality

$$q_x - 2g_{n+3} + q_x - 1g_{n+2} + q_x g_{n+1} = 0,$$

where $q_x - 2$, $q_x - 1$ and $q_x$ are some polynomials in Stiefel–Whitney classes $w_2$ and $w_3$, and the dimension of $q_x - i$, $i = 0, 1, 2$, is equal to $x - i$ (of course, $q_x - i = 0$ if $x < i$). In order to finish the proof, as in [12, p. 55], we are left to prove that $q_x - 2 = q_x - 1 = q_x = 0$, where $q_x - 2$, $q_x - 1$ and $q_x$ are interpreted as elements of the polynomial algebra $\mathbb{Z}_2[w_2, w_3]$ (since that will mean that all coefficients in the starting linear combination vanish).

So, from now on, $q_x - 2$, $q_x - 1$, $q_x \in \mathbb{Z}_2[w_2, w_3]$. Note that (8) holds in $\mathbb{Z}_2[w_2, w_3]$ as well. This is due to Theorem 2.1, since for a monomial $w_3^i w_3^j$ of the left-hand side in (8) one has $2(b + c) \leq 2b + 3c = n + 1 + x \leq 2n$ (since $x \leq n - 1$).

Let $s = n - 3 - 2^{i-1}$ (i.e., $n = 2^{i-1} - 3 + s$) and

$$(p_{x+s-1} \ p_{x+s} \ p_{x+s+1}) = (q_x - 2 \ q_x - 1 \ q_x) A^{s+1}. \tag{9}$$

Note that $s \geq 0$ and that the (cohomological) dimensions of polynomials $p_{x+s-1}, p_{x+s}, p_{x+s+1} \in \mathbb{Z}_2[w_2, w_3]$ are $x - s - 1, x + s, x + s + 1$ respectively. Multiplying equality (9) with the column $(g_{2^t-1-1} \ g_{2^t-1-2} \ g_{2^t-1-3})^T$, by (7) (for $r = s+1$ and $i = n+3$) and (8), we obtain that

$$p_{x+s-1}g_{2^t-1-1} + p_{x+s}g_{2^t-1-2} + p_{x+s+1}g_{2^t-1-3} = 0 \tag{10}$$

in $\mathbb{Z}_2[w_2, w_3]$. Since $g_{2^t-1-3} = 0$ and the polynomials $g_{2^t-1-2}$ and $g_{2^t-1-1}$ are coprime (and nonzero), we conclude that $g_{2^t-1-2} | p_{x+s-1}$. If $p_{x+s-1} \neq 0$, then by comparing dimensions of $g_{2^t-1-2}$ and $p_{x+s-1}$, we have $2^{i-1} - 2 \leq x + s - 1 \leq 2^{i-1} - 2 - s + s = 1 = 2^{i-1} - 3$ (since $x \leq \delta - n \leq 2^t - 5 - n = 2^t - 5 - (2^{i-1} - 3 + s) = 2^{i-1} - 2 - s$). This contradiction proves that $p_{x+s-1} = 0$, and then from (10) it follows that $p_{x+s} = 0$ also (since $g_{2^t-1-2} \neq 0$).

Now, if we multiply identity (9) with the matrix $B^{s+1}$, we obtain that

$$(0 \ 0 \ p_{x+s+1})B^{s+1} = \begin{pmatrix} w_3^{s+1}q_{x-2} & w_3^{s+1}q_{x-1} & w_3^{s+1}q_x \end{pmatrix}, \tag{11}$$

since $A^{s+1}B^{s+1} = w_3^{s+1}I$. For a matrix $C$ over the ring $\mathbb{Z}_2[w_2, w_3]$, let the matrix $\overline{C}$ be defined as the reduction of $C$ modulo $w_3$ (that is, each entry of $\overline{C}$ is the reduction of the corresponding entry of $C$ modulo $w_3$). Then, it is easy to check that

$$\overline{B^r} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ w_2 - 1 & 0 & w_2 \end{pmatrix}$$

for all $r \geq 1$. The reduction $\mathbb{Z}_2[w_2, w_3] \to \mathbb{Z}_2[w_2, w_3]/(w_3) \cong \mathbb{Z}_2[w_2]$ is a ring homomorphism, and so, $\overline{B^r} = \overline{B^r}$, $r \geq 1$. This means that the low-right entry of the matrix $B^{s+1}$ is equal to
$w_2^{s+1} + w_3\tilde{p}$ for some $\tilde{p} \in \mathbb{Z}_2[w_2, w_3]$. By (11), it follows that $w_3^{s+1}q_x = p_{x+s+1}(w_2^{s+1} + w_3\tilde{p})$, and so, $w_3^{s+1} | p_{x+s+1}$. If $p_{x+s+1} \neq 0$, then by comparing dimensions of $w_3^{s+1}$ and $p_{x+s+1}$, we obtain that $3(s+1) \leq x + s + 1 \leq 2s + 1 + s + 1 = 3s + 2$ (since $x \leq \delta - n \leq 3n - 2t + 7 - n = 2n - 2t + 7 = 2t - 6 + 2s - 2t + 7 = 2s + 1$). This contradiction implies that $p_{x+s+1} = 0$. Finally, by (11) we have that

$$q_{x-2} = q_{x-1} = q_x = 0,$$

and the proof is completed. \hfill \Box

**Remark 3.5** In \cite{7} Korbaš and Rusin proved that if $1 \leq s \leq 6$ and $2t-1 + \lceil \frac{s-1}{2} \rceil + 1 \leq n+3 \leq 2t - s - 3$, then $\text{charrank}(\tilde{\gamma}_{3,n}) \geq n + s + 1$. They also conjectured \cite[Conjecture 3.3]{7} that this is true for all $s \geq 1$. Proposition 3.4 (Theorem 1.1) confirms this conjecture. Indeed, since $\lceil \frac{n+1}{2} \rceil \geq \frac{n+2}{2}$, we have that $2(n+3) \geq 2t + 2 \cdot \lceil \frac{s+1}{2} \rceil + 2 \geq 2t + 2 \cdot \frac{s+2}{2} + 2 = 2t + s$, which implies $3n - 2t + 7 \geq n + s + 1$. Also, $n + 3 \leq 2t - s - 3$ implies $2t - 5 \geq n + s + 1$, and therefore, we have that

$$\text{charrank}(\tilde{\gamma}_{3,n}) \geq \min\{3n - 2t + 7, 2t - 5\} \geq n + s + 1.$$

**Remark 3.6** The characteristic rank of a smooth connected manifold was introduced in \cite{5}, and it is actually the characteristic rank of the tangent bundle over the manifold. If $n$ is even, then it is a known fact that all Stiefel–Whitney classes of (the tangent bundle over the) Grassmanian $\bar{G}_{3,n}$ are polynomials in Stiefel–Whitney classes of the canonical bundle $\tilde{\gamma}_{3,n}$, and vice versa (see \cite[p. 72]{5}). Therefore, by Theorem 1.1, for even $n$ and $t \geq 3$ such that $2t-1 \leq n + 3 < 2t$ we have that

$$\text{charrank}(\bar{G}_{3,n}) = \text{charrank}(\tilde{\gamma}_{3,n}) = \min\{3n - 2t + 7, 2t - 5\}.$$  


**4 Proof of Theorem 1.2**

The following theorem of Naolekar and Thakur gives an upper bound for the $\mathbb{Z}_2$-cup-length in terms of characteristic rank.

**Theorem 4.1** \cite{9} Let $M$ be a connected closed smooth $d$-dimensional manifold. Let $\alpha$ be a real vector bundle over $M$ and let $j \leq \text{charrank}(\alpha)$ be an integer such that every monomial $w_{i_1}(\alpha) \cdots w_{i_s}(\alpha), 1 \leq i_1 \leq \cdots \leq i_s \leq j$, in dimension $d$ vanishes. Then

$$\text{cup}(M) \leq 1 + \frac{d - j - 1}{r},$$

where $r$ is the smallest positive integer such that $H^r(M) \neq 0$.

In the case $M = \bar{G}_{3,n}$, we have that $d = 3n$ and $r = 2$. It is also a well known (and easily seen) fact that the nonzero class in $H^{3n}(\bar{G}_{3,n}) \cong \mathbb{Z}_2$ is not a polynomial in Stiefel–Whitney classes of the canonical bundle $\tilde{\gamma}_{3,n}$ (this can be seen, for instance, from the Gysin sequence (3): obviously, the map $H^{3n}(G_{3,n}) \xrightarrow{w_1} H^{3n+1}(G_{3,n}) = 0$ is not a monomorphism, and so, $p^* : H^{3n}(G_{3,n}) \to H^{3n}(\bar{G}_{3,n})$ is not onto). Therefore, for the bundle $\alpha := \tilde{\gamma}_{3,n}$ we can take $j := \text{charrank}(\tilde{\gamma}_{3,n})$, and then Theorem 4.1 gives us the inequality

$$\text{cup}(\bar{G}_{3,n}) \leq 1 + \frac{3n - \text{charrank}(\tilde{\gamma}_{3,n}) - 1}{2}. \quad (12)$$
The equality \( \cup(\tilde{G}_{3,2t-1-4}) = 2^{t-1} - 3, \ t \geq 4 \), was proved in [2] and, independently, in [5]. If \( 2^{t-1} - 3 \leq n \leq 2^{t-1} - 4 + \frac{2t-1}{3} \) (for some \( t \geq 4 \)), then by multiplying the second inequality with 3 we obtain that \( 3n - 2^t + 7 \leq 2^t - 5 \). Hence, by Theorem 1.1, \( \text{charrank}(\tilde{G}_{3,n}) = 3n - 2^t + 7 \), and from (12) it follows that

\[
\cup(\tilde{G}_{3,n}) \leq 1 + \frac{3n - (3n - 2^t + 7) - 1}{2} = 2^{t-1} - 3.
\]

The opposite inequality \( \cup(\tilde{G}_{3,n}) \geq 2^{t-1} - 3 \) holds by [7, (13)]. So, we have proved part (a) of Theorem 1.2.

For part (b), let \( 2^{t-1} - 4 \leq n \leq 2^{t-1} - 4 + \frac{2t-1}{3} \) (for some \( t \geq 4 \)). We know that \( w_2(\tilde{\gamma}_{3,2t-1-4})2^{t-1-4} \neq 0 \) (by [2, Corollary 4.12]), and for \( n \geq 2^{t-1} - 3 \), the fact \( w_2(\tilde{\gamma}_{3,n})2^{t-1-4} \neq 0 \) was proved in [7, p. 83].

Suppose that the class \( w_2(\tilde{\gamma}_{3,n})2^{t-1-3} \) is nonzero. Then, by the Poincaré duality there exists a class \( \sigma \in H^{3n-2^t+6}(\tilde{G}_{3,n}) \) such that \( \sigma \cdot w_2(\tilde{\gamma}_{3,n})2^{t-1-3} \neq 0 \), and, since \( 2^t - 6 \leq 2n + 2 < 3n \), we have that \( \cup(\tilde{G}_{3,n}) \geq 2^{t-1} - 2 \), which contradicts part (a) of the theorem. Therefore, \( w_2(\tilde{\gamma}_{3,n})2^{t-1-3} = 0 \).

We conclude that \( \text{ht}(w_2(\tilde{\gamma}_{3,n})) = 2^{t-1} - 4 \), which finishes the proof of Theorem 1.2.

References
