Constructions, Instructions, Interactions*

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Abstract

In this paper we consider geometrical construction problems and their role in mathematical education. Owing to the main features of construction problems (accuracy in making conclusions, strict structure, rigorous language and constructivistic nature), they can be fruitfully used in teaching mathematics and computer science. We shall try to shed a new light on construction problems, and show how they can be used in exercising mathematical rigor, but also for understanding and illustrating a range of mathematical and computer science theories. For these tasks, we emphasize the motivating role of software tools that can be used for interactive work. One such tool (WinGCLC) will be briefly presented.

1 Introduction

> *We see by experience that among equal minds and all other things being equal, he who possesses geometry, conquers and acquires an entirely new rigor.*
>
> B. Pascal, Pensées et opuscules,
> Hachette, p. 165, Note

In ancient Greece, geometry played an important role in general education and it always had its place, even when schooling consisted of only three or four subjects. Thanks to the Arabs, the knowledge of geometry again reached Europe in the Middle Age. Starting from the mid-nineteenth century (from Lobachevskii all the way to Hilbert), geometry had a fundamental role in the

*This work was partly supported by Serbian Ministry of Science and Technology research grant 1646.

reform of mathematics, leading to the introduction of strict concepts of language and axiomatization. This type of experience was largely responsible for the reforming other disciplines of mathematics, too. In the twentieth century, mostly as the result of the work of the Bourbaki group, the influence of the algebraic point of view on geometry was growing. The appearance of computers introduced new perspectives and applications in the field of geometry. During all that time, construction problems had an important role, and we believe that this role has gained a new meaning in the modern time characterized by the development of new mathematical and computer science concepts and theories.

In schools and at universities, geometry is studied to various extents, thus often serving for illustrating the axiomatic approach, as well. Although the study of geometry has lately been algebrized, we still believe that there is some space in which the synthetic approach can be illustrative and fruitful for understanding not only geometrical problems but also some other — wider mathematical and even computer science problems, concepts and ideas. To that sense, construction problems are extremely suitable because they are rigorous, but at the same time also more intuitive (since the level of abstraction is different, higher than geometrical axiomatic level and, hence, more easily comprehensible), as well as connected with the need for effective procedures and therefore more attractive to the students. Construction problems require accuracy in making conclusions, strict structure, rigorous language and constructive consideration (since it is not sufficient to prove the existence of a solution). That is why we consider that construction problems can play an important role in present-day education and that, at the same time, they represent an excellent field for training the students in such work. In further text we shall illustrate by a prototype example the questions of relevance to the formulation of a problem and components of the solution, including the most frequent mistakes. We shall demonstrate the way in which the construction itself represents an effective procedure which can be considered as a form of programming. This will be presented in a way which can be directly used in teaching, not only of geometry but of some other fields as well.

The study of geometry and construction problems also represents a suitable field for interactive teaching which can nowadays be supported by software tools, as well. Such interactive work is attractive and motivating for students, who can practically at any moment receive the feedback from the tools they use. Moreover, in that way geometry and construction problems are naturally studied in a manner appropriate to the contemporary science and technology. All these aspects and a multitude of possible links with the new mathematical and computer science disciplines shed a new light upon geometry and construction problems. In further text we shall present in brief one of such tools — WinGCLC.

Construction problems can also serve as a step towards computer science, not only regarding the fields such as graphics, but even as a link with theories such as the theory of computability, which will be seen in further text. We believe and try to demonstrate that there are reasons for analyzing construction problems, and we also point to the facilitating and motivating role of interactive work, especially presented by applying WinGCLC.

Overview of the paper: The rest of the paper is organized as follows: in Section 2 we discuss the main characteristics of formal constructions and geometrical construction problems; in Section 3 we give one example, worked out in full details; in Section 4 we stress some common mistakes in stating and solving geometrical construction problems and give some general suggestions; in Section 5 we discuss the software package WinGCLC — a tool for teaching and studying geometry; in Section 6 we discuss links between geometry and construction problems with a range of mathematical and computer science fields; in Section 7 we discuss how construction problems can be promoted and used in education; in Section 8 we briefly discuss related work and in Section 9 we draw final conclusions.

2 Formal geometrical constructions

The concept of formal geometrical constructions has been studied literally for thousands of years, and since the ancient Greeks (and especially since Euclid's *Elements* [7]), it has become a standard part of virtually any sort of education. One of the reasons for this was a general opinion that the rigor of geometrical constructions substantially helps in developing the process of logical thinking. The rigor in geometrical proofs was one of the motivating reasons for the modern reform of geometry, with Hilbert's *Grundlagen der Geometrie* as one of the milestones [9]. Modern approach to classical, synthetical geometry is still very much based on Hilbert's visions. During all this time, geometrical construction problems have remained one of the most rigorous and yet most attractive parts of geometry.

A geometrical construction is a sequence of specific, primitive construction steps. These primitive construction steps (which we also call *elementary constructions*) are based on using a *ruler* (or a *straightedge*[1]) and a *compass*, and they are:

- construction (with a *ruler*) of a line such that two given points belong to it;
- construction of a point which is an intersection of two lines (if such a point exists);
- construction (with a *compass*) of a circle such that its center is one given point and such that the second given point belongs to it;
- construction of a segment connecting two points;
- construction of intersections between a given line and a given circle (if such points exist).

[1]The term "straightedge" is sometimes used instead of "ruler" in order to emphasize that no markings which could be used to make measurements are allowed.

Note that the above primitive constructions use *abstract* instruments: a *ruler* and a *compass*. For Euclidean geometry, a ruler and a compass, as the usual real-world instruments, can help in making approximative representation of formal constructions in Cartesian plane. However, one should not mix-up these instruments with the abstract instruments whose application in different geometries can have different properties (e.g., an abstract ruler will differ in Euclidean and hyperbolical geometries). Also, it cannot be overemphasized that there is a need to distinguish the abstract (i.e., formal, axiomatic) nature of geometrical objects from their usual interpretations (e.g., Cartesian interpretations). A geometrical construction is a mere procedure consisting of abstract steps and not a picture. However, for each construction, there is its counterpart in the standard Cartesian model.

By using the set of primitive constructions, one can define more complex constructions (e.g., the construction of a right angle, a construction of the midpoint of a line segment, a construction of the bisector of an angle, etc.). In describing geometrical constructions, both primitive and higher level constructions and used.

The solution of a geometrical construction problem traditionally includes the following four phases/components: analysis, construction, proof and discussion (see, for instance, [10]). Each of them has its important role in the solution as a whole.

Solving of geometrical construction problems requires great logical accuracy as well as precise language. Although these properties cannot, on their own, make the solution correct, they not only increase its value, but also correctly direct a way of thinking both of the person writing the solution, on one hand, and the person reading it, on the other.

2.1 Analysis

Generally, in analysis one starts from the assumption that a certain geometrical object satisfies the conditions of the problem (conditions Γ) and proves that properties Λ enabling the construction are then also satisfied. The analysis is correct only if it contains all the proved implications which make complete construction possible. For instance, if it is necessary to construct the triangle ABC which satisfies certain properties, then the analysis must contain proved properties which are fulfilled for points A, B, and C, thus enabling the effective construction of these points.

2.2 Construction

Construction is based on the analysis, that is, on the properties Λ which are proved in it. If the analysis is carried out sufficiently precisely, then all the steps of the construction are directly linked with the assertions in the analysis. However, it is assumed in the analysis that a certain figure satisfies the properties of the problem, but nothing is constructed within the analysis (e.g., if an "auxiliary" point P is used in the analysis, it is not said that "... we construct point P which is ...", but for example "... let point P be such that ..."). Construction, on the other hand, is a constructive process and the only way to use in it a certain (auxiliary) object is to construct it first (and then we say "... we construct the object ...").

The nature of construction is formal and it is, in fact, the description of the procedure, that is, the set of formal and abstract steps (based on the use of abstract instruments such as a ruler and a compass), and not a picture (to be obtained, for instance, on the paper by using a ruler and a compass). Such a picture can facilitate the understanding of the description of the construction, but it can in no way replace one.

In constructions one must not use misleading and wrong phrases such as: "let us draw a straight line", "let us draw a segment", "let us draw a normal", "let us form a circle", "let us circumscribe a circle around a triangle", "let us encompass in the opening of the compass", "let us rotate the triangle ABC in the clockwise direction", "let us denote by B the point which is up/down/above/below/left/right (from) the point A", etc (see also Section 4).

2.3 Proof

In this part of the solution, it is necessary to prove that, if a figure is constructed on the basis of the given description (that is, if it satisfies conditions Λ), then it satisfies the conditions of the problem (i.e., conditions Γ). Within the framework of the construction problem, proof is one of the places where most of the mistakes are made, including the standard ones:

- wrongly established goal; a frequent error consists of unnecessarily proving unnecessary facts, while omitting what is necessary: the proof that the constructed figure satisfies the properties of the problem by the very statement of the problem;

- incorrect referring to analysis ("the proof is analogous to the analysis"); the analysis and the proof represent, basically, two directions of one equivalence and by rule are essentially different, so they can almost never (or literally never) be analogous;

- incorrect referring to proved conjectures (one of the most frequent errors; for instance, in order to apply the assertion $PP_a = b - c$ from (the following) Section 3, we must, first of all, ascertain that the points P and P_a obtained in the construction are indeed the touching points of the inscribed circle and the escribed circle of the triangle ABC with the line BC).

2.4 Discussion

In the discussion, it is considered how many possible solutions to the problem there exist. Ideally, the number of solutions should be expressed effectively in the function of mutual relations of the given elements, but sometimes it is sufficient to express it implicitly in the function of the relation of the figures obtained during the construction.

3 Worked example

In this section we shall give a fully detailed solution to one geometrical construction problem. The problem is of moderate difficulty level and we have chosen it in order to illustrate several important issues and to demonstrate the required rigour of reasoning. The problem requires that a triangle should be constructed, given the difference between its two sides, one median line and the radius of its inscribed circle. In order to solve it, we first need to prove two auxiliary conjectures and to solve one auxiliary construction problem.

We denote by $\mathcal{B}$ a primitive relation *between* of arity 3, which can also be generalized (defined) to arity n, $n > 3$. We denote by $\cong$ a primitive *congruence* relation. We denote points by capital Latin letters and we denote lines, segments and segments' measures by small Latin letters; we denote angles and their measures by small Greek letters.

Lemma 1 *If the circle with the centre at the point O touches the sides of the angle $\angle XYZ$ at points X and Z, then $YX \cong YZ$ holds.*

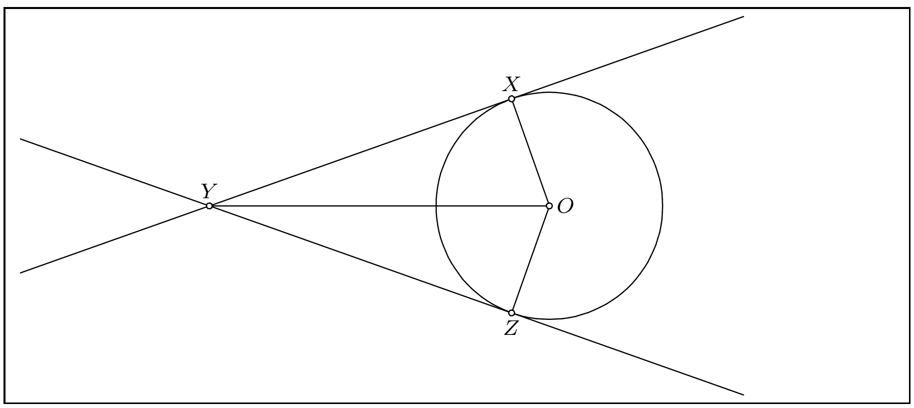

Figure 1: Illustration for Lemma 1

Proof:

Since $OX \cong OZ$, $OY \cong OY$ and $\angle OXY = \angle OZY = \frac{\pi}{2}$, it follows that the triangles OXY and OYZ are congruent, and therefore $YX \cong YZ$. QED[2]

Lemma 2 *If S is the centre of the circle inscribed in the triangle ABC (for which $AC \geq AB$ holds), A_1 the midpoint of the side BC, P the tangent point of*

[2]QED — Quod errat demonstrandum (*what was to be proved*), since the middle Age the proofs have been concluded by this Latin sentence.

the line BC with the inscribed circle, P_a the tangent point of the line BC with the escribed circle which corresponds to the vertex A, and P' the point symmetric to the point P with respect to the point S, then the following holds: $\mathcal{B}(A, P', P_a)$, the point A_1 is the midpoint of the segment PP_a, and $PP_a = AC - AB$.

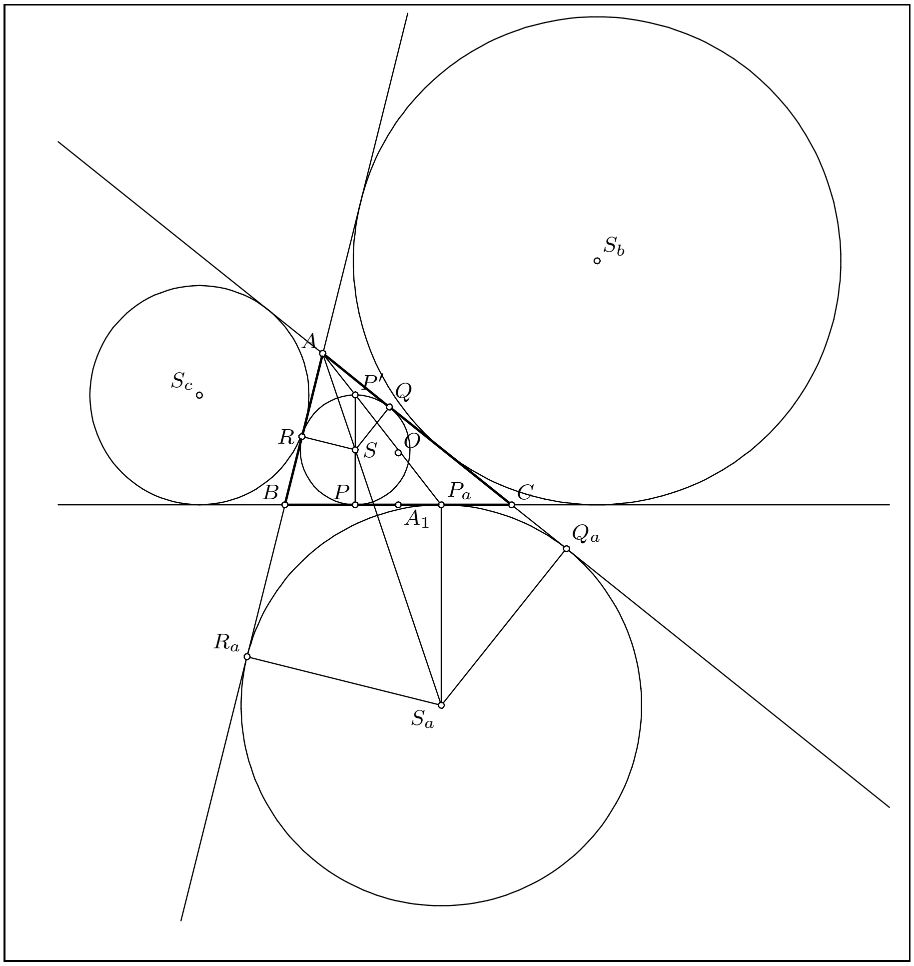

Figure 2: Illustration for Lemma 2

Proof:

- If $AB \cong AC$ holds, then the line AS is the bisector of the side BC and it contains the points P, P' and P_a. Moreover, the points A_1, P and P_a are identical and, hence, $PP_a = AB - AC$, as claimed.
- If $AB \cong AC$ does not hold, then the points P and P_a are not identical and the lines AS and $P'P_a$ are different. Let Q be the tangent point of

the line AC with the inscribed circle and Q_a the tangent point of the line AC with the escribed circle which corresponds to the vertex A. Let $\hat{P}$ be the intersection point of the lines SP and AP_a. Since $SP \perp BC$ and $S_aP_a \perp BC$, it follows that $S\hat{P} \| S_aP_a$, and therefore, on the basis of the Thales' Theorem, it holds that $S\hat{P} : S_aP_a = AS : AS_a$. Since $SQ \perp AC$ and $S_aQ_a \perp AC$, it follows that $SQ \| S_aQ_a$, and therefore, on the basis of the Thales' Theorem, it holds that $SQ : S_aQ_a = AS : AS_a$. This means that $S\hat{P} : S_aP_a = AS : AS_a = SQ : S_aQ_a$, and then it follows that $S\hat{P} = SQ$ (because $S_aP_a = S_aQ_a$). Therefore, the point $\hat{P}$ lies on the circle inscribed in the triangle ABC and it lies on the line SP. It cannot be identical to the point P, because in that case the points P and P_a would be identical, and the following would hold: $AB \cong AC$, which is contrary to the assumption. Hence, the point $\hat{P}$ is symmetric to the point P with respect to the point S, that is, the points P' and $\hat{P}$ are identical, and then it follows that the point P' lies on the line AP_a. The point P' lies on the circle inscribed in the triangle, while the point P_a lies on the side BC, and therefore $\mathcal{B}(A, P', P_a)$ holds.

Let a, b and c be the lengths of the sides BC, AC and AB, respectively, and let $p = \frac{1}{2}(a+b+c)$. Let Q and R be the points in which the circle inscribed in the triangle ABC touches the lines AC and AB. Further, let Q_a and R_a be the touching points of the lines AC and AB with the escribed circle of the triangle ABC corresponding to the vertex A.

Let us prove that $BP = p - b$. On the basis of Lemma 1, it holds that $BP \cong BR$, $AR \cong AQ$ and $CP \cong CQ$. Moreover, it also holds that $\mathcal{B}(B, P, C)$, $\mathcal{B}(B, R, A)$ and $\mathcal{B}(A, Q, C)$, from which it follows that $BP = BC - CP$, $BR = BA - AR$ and $AC = AQ + CQ$, and therefore

$$BP = \frac{1}{2}(BP+BR) = \frac{1}{2}(BC-CP+BA-AR) = \frac{1}{2}(BC+BA-CQ-AQ) =$$

$$= \frac{1}{2}(BC + BA - AC) = \frac{1}{2}(a + c - b) = p - b\ .$$

Let us prove that $CP_a = p - b$. On the basis of Lemma 1, it follows that $AR_a \cong AQ_a$, $BP_a \cong BR_a$ and $CQ_a \cong CP_a$. The following also holds: $\mathcal{B}(A, B, R_a)$, $\mathcal{B}(A, C, Q_a)$ and $\mathcal{B}(B, P_a, C)$, therefore

$$AQ_a = \frac{1}{2}(AQ_a + AR_a) = \frac{1}{2}(AB + BR_a + AC + CQ_a) =$$

$$= \frac{1}{2}(AB + AC + BP_a + CP_a) = \frac{1}{2}(AB + AC + BC) = \frac{1}{2}(a + b + c) = p\ .$$

From $AQ_a = p$ it follows that $CQ_a = AQ_a - AC = p - b$. On the basis of Lemma 1, $CP_a = CQ_a$ holds, therefore $CQ_a = p - b$.

If it holds that $AC > AB$, then $\mathcal{B}(B, P, A_1, P_a, C)$ holds, so that $A_1P = BA_1 - BP = \frac{1}{2}a - (p - b)$ and $A_1P_a = A_1C - P_aC = \frac{1}{2}a - (p - b)$.

Hence, $A_1P = A_1P_a$, and therefore from $\mathcal{B}(P, A_1, P_a)$, it follows that the point A_1 is the midpoint of the segment PP_a and $PP_a = A_1P + A_1P_a = \frac{1}{2}a - (p - b) + \frac{1}{2}a - (p - b) = a - 2p + 2b = a - (a + b + c) + 2b = b - c$. (If it holds that $AB > AC$, it can be proved that $PP_a = c - b$, by analogy).

QED

Problem 1 *Construct the tangent through the point P touching the circle k.*

Analysis:

Let us suppose that t is the tangent through the point P touching the circle k. If such a tangent exists, the point P either lies outside the circle k or it lies on the circle k. Let O be the centre of the circle k.

- If the point P lies outside the circle k, then the angle PTO is a right angle, where T is the point of contact of the tangent t and the circle k. Therefore it follows that the point T is the intersecting point of the circle k and the circle whose diameter is the segment PO. The line t is determined by the points P and T.
- If the point P lies on the circle k, then the tangent t is normal to the line PO.

Construction:

- If the point P lies outside the circle k, let us construct the circle l whose diameter is the segment OP. Let us denote by T one of the intersecting points of the circles l and k. Let us construct the line t determined by the points P and T. The line t is the tangent through the point P touching the circle k.
- If the point P lies on the circle k, let us construct the line t on which the point P lies, and which is perpendicular to the line OP. The line t is the tangent through the point P, touching the circle k at the point P.
- If the point P lies inside the circle k, there is no tangent through the point P touching the circle k.

Proof:

- If the point P lies outside the circle k, the point T, on the basis of the construction, lies on the circle whose diameter is the segment OP, from which it follows that the angle PTO is a right angle. The line t is determined by the points O and T, and therefore it is the tangent through P touching k, as claimed.
- If the point P lies on the circle k, the line t is, on the basis of construction, perpendicular to OP, and therefore it is the tangent through the point P touching the circle k.

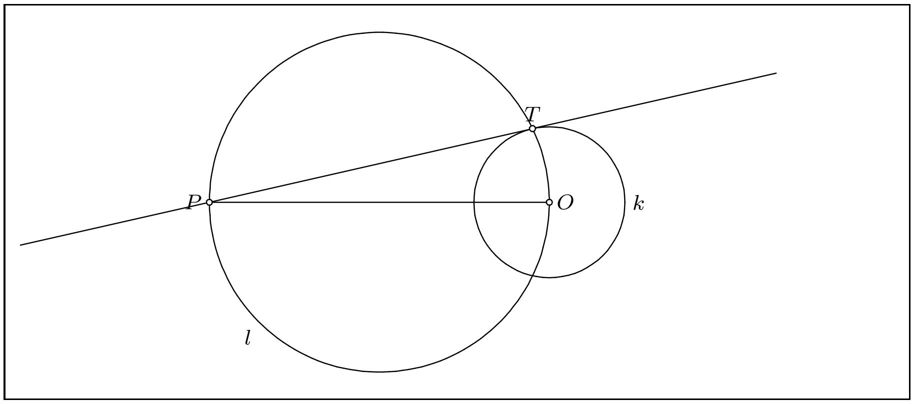

Figure 3: Illustration for Problem 1

Discussion:

- If the point P lies outside the circle k, then there are two tangents through P touching k.
- If the point P lies on the circle k, then there is one tangent through P touching k.
- If the point P lies inside the circle k, then there is no tangent through P touching k.

Problem 2 *Construct a triangle ABC such that $AC - AB = d$, the radius of its inscribed circle is congruent to r, and the median line that corresponds to A is congruent to t_a, where d, r and t_a are given.*

Analysis:

Let us assume that the triangle ABC fulfills the conditions of the problem. Let A_1 be the midpoint of BC, P the tangent point of the line BC with the inscribed circle, P_a the tangent point of the line BC with the escribed circle that corresponds to the vertex A, and P' the point symmetric to the point P with respect to the point S (which is the centre of the inscribed circle of the triangle ABC). On the basis of Lemma 2, the point A_1 is the midpoint of the segment PP_a and $\mathcal{B}(A, P', P_a)$ holds. Since $AA_1 \cong t_a$, the point A lies on the circle l with the centre A_1 and the radius t_a. The point A is the intersecting point of the circle l and the line P_aP'. The points B and C lie on the line PP_a and on the tangents through the point A touching the circle with the centre S and the radius SP.

Construction:

1. Let us construct the segment PP_a congruent to the given segment d.

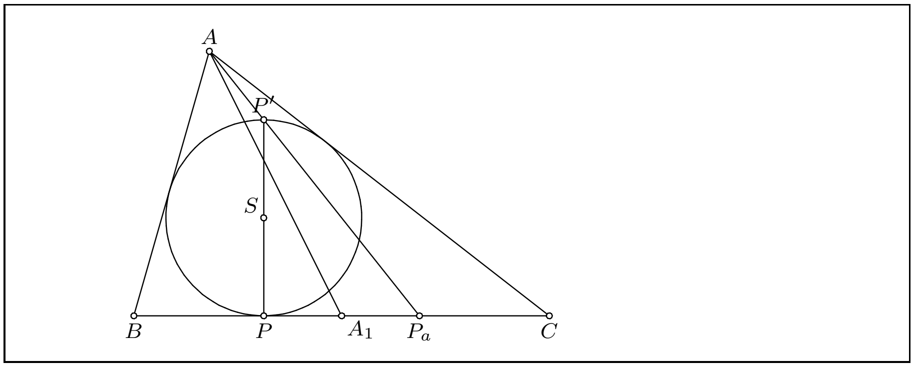

Figure 4: Illustration for Problem 2

2. Let us construct the perpendicular to the line PP_a at the point P.
3. On the constructed perpendicular, let us determine the point S such that $PS \cong r$ holds.
4. Let us denote by P' the point symmetric to the point P with respect to S.
5. Let us denote by A_1 the midpoint of the segment PP_a.
6. Let us construct the circle l with the centre A_1 and the radius t_a.
7. Let us denote by A the intersecting point of the circle l and the line $P'P_a$.
8. Let us construct the circle k with the centre S and the radius r.
9. If $\mathcal{B}(A, P', P_a)$ holds, then let us construct (on the basis of the auxiliary construction, see Problem 1) the tangents through the point A touching the circle k.
10. Let us denote the intersecting points of these tangents and the line PP_a by B and C.

If $\mathcal{B}(A, P', P_a)$ holds, the triangle ABC fulfills the conditions of the problem.

Proof:

On the basis of the construction, the lines BC, AB and AC touch the circle k, therefore the circle k with the radius r is the inscribed circle of the triangle ABC.

The point S is the centre of the inscribed circle of the triangle ABC, while the point P is its touching point with the side BC. The point P' is, on the basis of the construction, symmetric to the point P with respect to the point S. On the basis of Lemma 2, the lines AP' and BC intersect at the point which is the tangent point of the line BC with the escribed circle of the triangle that corresponds to the vertex A. On the basis of the construction, the point P_a is the intersection point of the lines AP' and BC, therefore P_a is the tangent point of the line BC with the escribed circle of the triangle that corresponds to the vertex A. On the basis of Lemma 2, $PP_a = AC - AB$ holds, while on the basis of the construction $PP_a \cong d$ holds, therefore it follows that $AC - AB = d$.[3]

The points P and P_a are the tangent points of the line BC with the inscribed circle and the escribed circle (that corresponds to the vertex A), respectively. On the basis of Lemma 2, the midpoint of the segment BC is the midpoint of the segment PP_a. On the basis of the construction, A_1 is the midpoint of PP_a, hence it follows that A_1 is the midpoint of the segment BC. On the basis of the construction, the point A lies on the circle with the centre A_1 and the radius t_a, from where it follows that $AA_1 \cong t_a$, which we wanted to prove.

Therefore, the constructed triangle ABC fulfills the conditions of the problem.

Discussion:

On the basis of the analysis it follows that in the triangle which fulfills the conditions of the problem $\mathcal{B}(A, P', P_a)$ holds, hence the condition $AA_1 > A_1P'$ must also hold, as well as the following:

$$t_a > \sqrt{PP'^2 + PA_1^2} = \sqrt{(2r)^2 + \left(\frac{d}{2}\right)^2}.$$

The solution exists if and only if this condition is fulfilled and then the solution is unique up to congruence.

4 DOs and DON'Ts

The worked example (Section 3) illustrates some of criteria important both in stating and in solving construction problems. Here are some DOs and DON'Ts:

[3]We use the symbol $\cong$ to denote the congruence of the segments, and the symbol $=$ to denote the measure of the segments. Depending on the context, it is sometimes more suitable to use the former, and sometimes the latter symbol.

DON'Ts	DOs
Never state the problem this way: "Construct a triangle ABC if $b-c, r, t_a$ are given".	Make a self-contained statement. In place of the confusing $b-c$ (b and c are measures, so $b-c$ cannot be a line segment), introduce a new line segment and state the condition in terms of vertices A, B, C or defined objects (e.g., "... such that $AC - AB = d$").
Never replace a construction by a figure. Never refer to a figure (either in the statement, or in the solution).	Provide a figure, but keep in mind that it is just an illustration, not a part of the solution.
Never refer to a proved conjecture without caution. (For instance, in the worked example (Section 3) one must not use $PP_a = b - c$ before it has been proved that P and P_a meet the conditions of Lemma 2.)	Prove that the constructed object meets the conditions of a lemma before using the lemma.
In the construction never refer or use the objects that have not yet been constructed (even the corresponding objects were defined in the analysis). (For instance, in the worked example one cannot say: "Denote by B and C the intersection points of these tangents with the line BC".)	Always define all objects being used in the construction. The only objects available (defined) at the beginning of the constructions are the given objects (in the worked example: d, r, t_a).
Never use terms like "let us construct..." in other parts of the solution apart from the construction.	In the construction, use terms like "Let us construct..." or "Let us denote by", while in other parts terms like "Let ... be a/an" should be used.

5 WinGCLC and interactive support

WinGCLC package[4] [13] is a tool for teaching and studying geometry, and especially studying construction problems. Students can make different attempts in making constructions and/or exploring some geometrical properties. Interactive work makes these attempts more interesting and more fruitful. This

[4] WinGCLC is freely available on-line from `www.matf.bg.ac.yu/~janicic/gclc/`. The mirrored version is available from EMIS (The European Mathematical Information Service) `www.emis.de/misc/index.html`. There exists a Windows version, as well as command-line versions GCLC for DOS and for LINUX. The name GCLC is derived from *Geometry Constructions ⇒ LaTeX Converter*.

way of solving construction problems helps computer science students to better understand geometry notions and vice versa.

Constructions for a certain problem are described in GCLC language. These descriptions are compiled by the WinGCLC processor (and can be exported to different output formats, such as LaTeX format or the bitmap format). It provides an easy-to-use support for all primitive constructions, but also for a range of higher-level constructions. In addition, WinGCLC provides support for isometric transformations, general conics, etc.[5]

While a construction is an abstract procedure, in order to make its representation in Cartesian plane (or, more precisely, in Cartesian model of Euclidean plane), we still have to make some link between these two. For instance, given three vertices of a triangle we can construct a center of its inscribed circle (by using primitive constructions), but in order to represent this construction in Cartesian plane, we have to take three particular Cartesian points as vertices of the triangle. Thus, figure descriptions in WinGCLC are usually made by a list of definitions of several (usually very few) fixed points (defined in terms of Cartesian plane, i.e., by pairs of coordinates) and a list of construction steps based on these points.

The example given in Fig. 5 illustrates one simple geometrical construction. Groups of commands are explained in comments within the description itself. The output (in LaTeX format) is presented in Fig. 6. In this example we construct three bisectors of the angles of a triangle ABC. It is very well-known that these three lines intersect at one point (at the center of the inscribed circle). This simple property (as well as much more complex properties or hypotheses) can be, in a sense, explored and investigated by WinGCLC. Namely, `d` is the distance between `S` and `S'`, and one can use the so called "watch window" (or "geometry calculator") to check that `d`$= 0$ (for these and other particular vertices).

As an example, we also present the construction for Problem 2 (d is given as D_1D_2, r is given as R_1R_2, and t_a is given as T_1T_2). Figure 7 presents the GCLC description, Figure 8 presents the LaTeX output and Figure 9 shows the corresponding WinGCLC screenshot.

WinGCLC can also serve as a tool for making digital illustrations. It is guided by the idea of geometrical construction problems and by the idea of "describing figures" rather than of "drawing figures" (so, in a sense, this system is in spirit close to the TeX/LaTeX system [15, 16], or it is parallel to it). There is an interface which enables simple and interactive use of a range of functionalities, including making animations, "watch window", interactive changing of a figure description, etc.

[5] Although motivated by the formal geometrical constructions, WinGCLC provides support for some non-constructible objects, too (for instance, in WinGCLC it is possible to determine/use a point obtained by rotation for 1°, although it is not possible to construct that point by a ruler and a compass).

```
% fixed points
point A 5 5
point B 50 5
point C 20 50

% determining bisectors
bis a B A C
bis b A B C
bis c A C B

% determining intersections of bisectors
intersec S a b
intersec S' a c
distance d S S'

% marking points
cmark_b A
cmark_b B
cmark_t C
cmark_t S

% drawing sides of the triangle ABC
drawsegment A B
drawsegment B C
drawsegment C A

% drawing the circle inscribed in ABC
line a1 B C
perp a2 S a1
intersec P a1 a2
drawcircle S P
```

Figure 5: Example 1 — a GCLC description of a construction

6 Links

In this section we discuss several interesting and important links between geometrical construction problems and some other mathematical or computer science fields (this overview is brief and simplified). These links should be useful for better understanding of both their sides. They shed a new light on some fields and give a unifying view even for some areas that seem not very close at the first glance. We believe that these links provide evidence and supply a range of ideas for potential multiple usefulness of studying construction problems (especially at the university level of education).

6.1 Constructions as programs

Due to their constructive, effective nature, geometrical constructions are, in a sense, similar to programs. As we have already said, a construction is not a picture, but a procedure of abstract construction steps. The language of constructions can be seen as one programming language (although such language

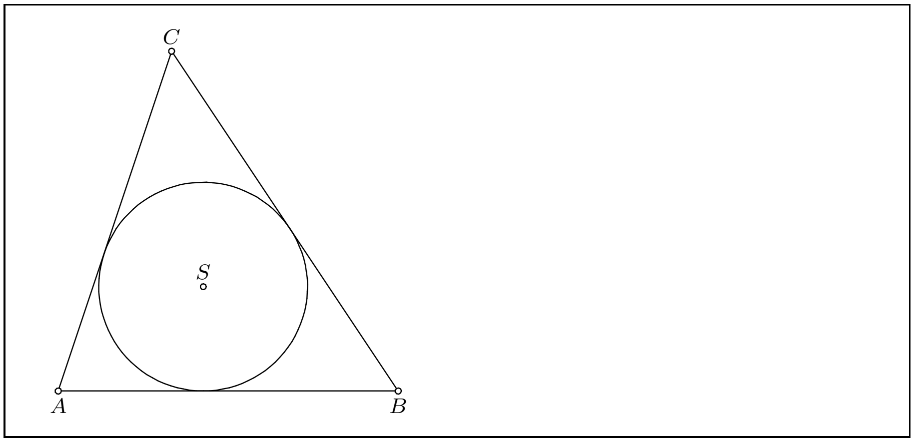

Figure 6: Output for Example 1

does not need to be very expressive as constructions usually do not require devices like loops and arrays). Geometrical construction problems can help students skilled in programming to understand geometry, while they can also help students acquainted with geometry and construction problems to understand programming, and the above-said applies to all education levels. More on representing geometric constructions as programs see in [23, 11].

We believe that this link is very fruitful in education, while the benefits are multiplied and increased if some geometry interactive/dynamic software tools are used (e.g., WinGCLC (Section 5), Cinderella [19] etc.).

6.2 Construction problems and constructive type theory

Martin Löf's constructive type theory [17] is one of the most interesting and the most important new theories which try to unify large parts of mathematics and computer science. In this theory, these are some of the most important principles: *a set is a theorem*, *an element of a set is a proof of a theorem*, *a specification is a theorem*, *a program is a proof of a theorem*. Without going into more details, we could add the following to these principles: *a statement of a construction problem is a specification*, *a construction is a program*. This unifying view/illustration (with all its meanings and consequences) can be interesting and useful to the university level of education.

6.3 Abstract constructions and their link with the object-oriented paradigm

Absolute geometry does not have an axiom on parallel lines and thus is incomplete. If we add Euclid's axiom on parallel lines, we obtain Euclidean geometry, while if we add the hyperbolical axiom on parallel lines we obtain hyperbolical geometry. Conjectures valid in absolute geometry are valid in both Euclidean

```
dim 120 85

% given objects
point D1 5 80
point D2 20 80
drawsegment D1 D2
midpoint D D1 D2
printat_b D {d}

point R1 5 75
point R2 17 75
drawsegment R1 R2
midpoint R R1 R2
printat_b R {r}

point T1 5 70
point T2 45 70
drawsegment T1 T2
midpoint T T1 T2
printat_b T {t_a}

area 5 10 110 60

% step 1
point P 50 5
translate P_a D1 P D2
cmark_b P
cmark_b P_a
line p P P_a
drawline p

% step 2
perp s P p

% step 3
translate S' R1 P R2
circle m P S'
intersec2 S S'' s m
cmark_l S

% step 4
sim P' S P
cmark_rt P'

% step 5
midpoint A_1 P P_a
cmark_b A_1

% step 6
translate T' T1 A_1 T2
circle l A_1 T'

% step 7
line p' P_a P'
drawline p'
intersec2 A A' p' l
cmark_rt A
drawsegment A A_1

% step 8
circle k S P
drawcircle k

% step 9
midpoint O A S
circle o O A
intersec2 Tb Tc o k
line t' A Tb
line t'' A Tc
drawline t'
drawline t''

% step 10
intersec B p t'
intersec C p t''
cmark_lb B
cmark_b C
```

Figure 7: The GCLC description of the construction for Problem 2

geometry and hyperbolical geometry (while, of course, some conjectures are valid in Euclidean and not in hyperbolical geometry, and vice versa). Also, some abstract constructions may be the same in both Euclidean geometry and hyperbolical geometry. For instance, the construction of a regular triangle ABC given the points A and B can be described in the following way in both Euclidean and hyperbolical geometry: *construct a circle k_1, with center A and with B belonging to it, construct a circle k_2, with center B and with A belonging to it, denote by C an intersection of circles k_1 and k_2*. Note that the *construction of a circle* is in this description treated as an abstract step with different treatment in Euclidean and hyperbolical geometry. There is a link between this hierarchy and the object-oriented paradigm in programming. Absolute geometry can correspond to an abstract class, while Euclidean and hyperbolical geometry inherit absolute geometry and interpret its virtual methods. This link very naturally illustrates and helps understanding both its sides.

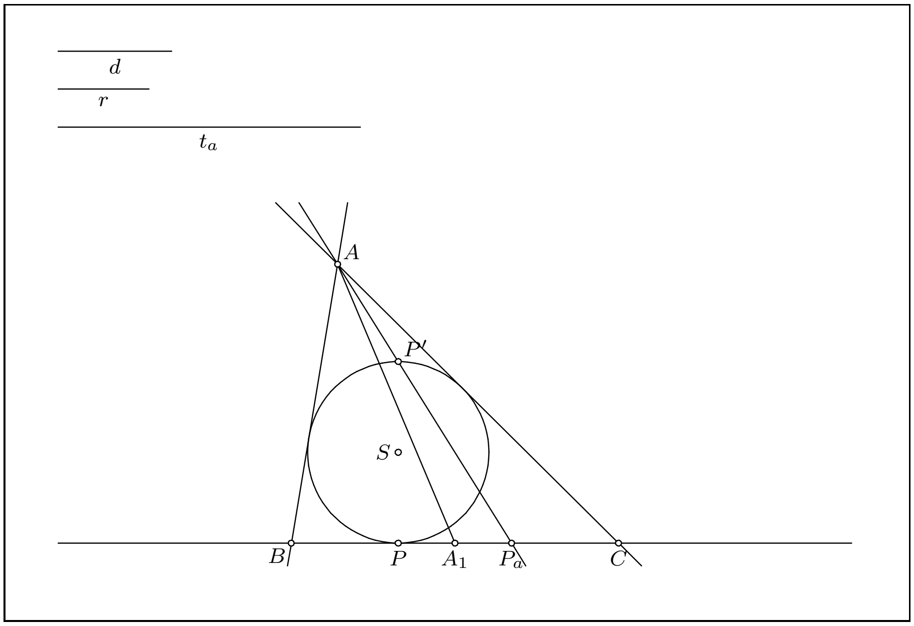

Figure 8: Output for the GCLC description of the construction for Problem 2

6.4 Construction problems and WinGCLC as a tool for explaining a notion of model

We have already drawn attention to the need to distinguish the abstract (i.e., formal, axiomatic) nature of geometrical objects from their usual models (e.g., Cartesian models). A geometrical construction is a mere procedure of abstract steps and not a picture. However, for each construction, there is its counterpart in the standard Cartesian model. On the other hand, a hyperbolical construction can be interpreted in Poincaré's disc model of hyperbolical plane, and further interpreted in Cartesian model of Euclidean plane. There exists WinGCLC module HYP-EUC[6] which transforms an abstract hyperbolical construction into an abstract Euclidean construction in Poincaré's disc model of a hyperbolical plane. The corresponding Euclidean construction (in GCLC) can then be processed in a usual way. By using this module, GCLC works as a platform for both Euclidean and hyperbolical geometry. It is interesting to investigate the same abstract construction in two geometries: Euclidean and hyperbolical one. These WinGCLC features enable understanding a notion of Poincaré's model, but also a general notion of model.

[6]Developed by Ivan Trajković (Faculty of Mathematics, University of Belgrade), under the supervision of the second author.

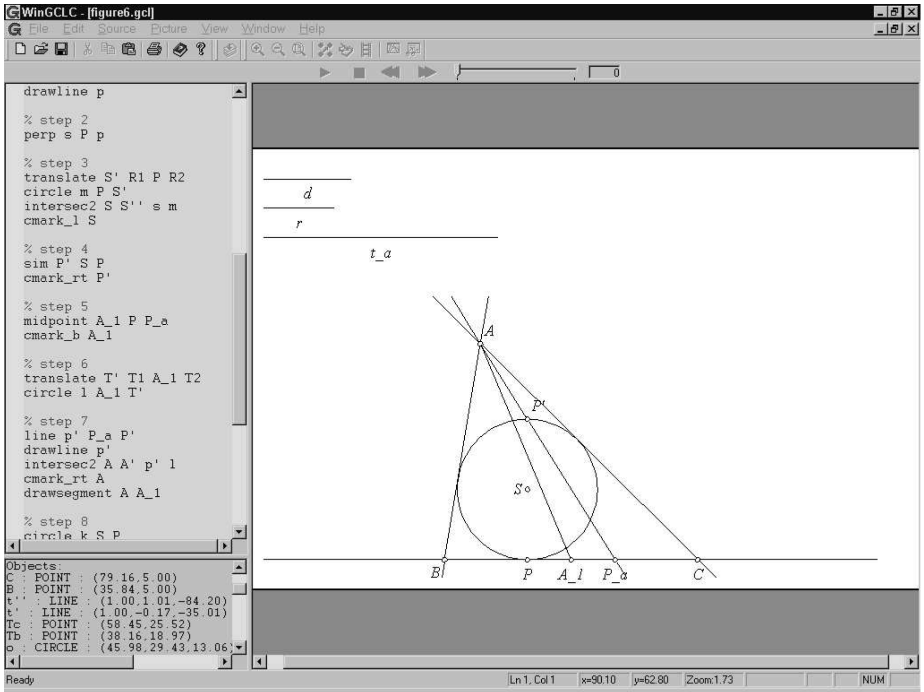

Figure 9: A screenshot of WinGCLC (with the description of the construction for Problem 2)

6.5 Construction problems and theory of computability

There is a number of formalisms for defining computable functions: Turing machine, Post machine, UR machine, lambda calculus, Markov's algorithms, recursive functions etc [6]. The theory of computability is fundamental for theoretical computer science and it is vital in many practical problems as well. It is interesting that computable functions can be, under some conventions, also defined and studied in terms of Euclidean constructions by a ruler and a compass within, so called, "Euclid's Abstract Machine" [5].

6.6 Construction problems and automated problem solving and theorem proving

Some fragments of geometry are decidable [24] and are very interesting domains for automated theorem proving. Gelernter's *Geometry machine* [8], written in mid-fifties, was one of the first theorem provers. It addressed fragments of constructive geometry and introduced a number of ideas and techniques later used in many systems. Chou's geometry theorem prover for the constructive part of geometry [4] based on Wu's algorithm (and algebraic properties) is considered by many as one of the most successful theorem provers in general. Chou and colleagues also made a theorem prover that produces traditional proofs of geometrical conjectures [3]. In [12] the authors described a theorem prover which proves conjectures down to the level of geometry axioms, and can be adapted to dealing with higher, construction steps (as a basic level, instead of the level of axioms). There also exist algorithms for solving some classes of construction problems (see, for instance, [2, 22, 21]). This field strongly links geometry and construction problems with automated reasoning and algorithmics.

7 Promoting construction problems in education

In this work we promote both using the construction problems for exercising rigorous mathematical reasoning and exploiting new technologies for better understanding of mathematical/geometrical ideas. Moreover, such technologies develope students' need for justification of relationships, improve their reasoning skills, support the link between inductive and deductive reasoning and encourage students to understand these two more deeply.

Advancing technologies and media shed new light on mathematics and enable substantial improvements in education. This is especially the case with geometry which can be almost often linked to some visualisations. Interactive/dynamic software tools motivate and help students to visualize geometry notions and statements (both those intuitive and counterintuitive). Technology in the form of interactive/dynamic geometry software packages provides powerful new means of studying geometry. As the value of this technology has become recognized, many teachers/professors have integrated it into their geometry courses (see, for instance, [1, 14]).

Within an active classroom environment, and especially within a work supported by suitable software tools, students can/should be encouraged to make conjectures that follow through a series of explorations. By using interactive software tools (such as, for instance, WinGCLC) students have an opportunity to explore and experiment, to change the set of initial figures or the construction procedure and consider the resulting effect. This way, students use inductive reasoning when they perform investigations and make conjectures about some geometrical properties or about ways for constructing some figures. After this stage, having some conjectures made (either some general geometrical statements, or some statements within a particular construction problem), students should be asked to provide a formal, deductive proof of the conjectures. In such a way, students improve both their inductive and deductive reasoning and also their understanding of the link between these two. This approach provides mathematical experiences that proceed from the simple to the complex and from the concrete to the abstract. Some of possible teaching examples (depending on the level of students' knowledge) are constructions of regular polygons, images of points by a circle inversion, constructions within Poincare's model of hyperbolical plane etc.

The example given in Section 3 provides a sample (presented in full details) of classroom applications that can be studied both as a rigorously solved geometrical problem and a problem that can be interactively explored by using software tools. Such tools stimulate various cognitive processes, actively engage students and give them deeper understanding of subject matters. Of course, creative teachers should be able to define other suitable classroom demonstrations and find many other attractive examples and ways of motivating students to investigate the subject interactively. We stress that students should not conceive geometrical construction problems as a mere collection of facts and procedures, but should also develop higher-level reasoning and understand meta-procedures for solving whole classes of problems and subproblems.

In addition to the above ideas, we believe that a substantial quality of education can be gained by permanent promoting links between geometry and construction problems with other mathematical and computer science notions and fields (some of these links were discussed in Section 6). These links should enable students better perceiving of horizontal relationships between fields which are not related at the first glance and to understand families of theories in a more compact, more uniform way.

8 Related work

The work presented here is, in one way, related to the long line of rigorous teaching and studying of construction problems, the line that goes back for several thousands years and is still actual (see, for instance, [10]).

On the other hand, this work and the presented tool WinGCLC are closely related to a family of modern software tools for interactive studying of geometry. One of the first such tools and one of the most popular in many classroom environments is *Geometer's Sketchpad* (see, for instance, [20]). Also, there are other popular tools, such as Cinderella [19] and JavaView [18], that have support for a range of geometries.

Finally, this work is also related to a number of mathematical and computer science fields with strong links to geometry and construction problems (as discussed in Section 6).

9 Conclusions

In this paper we have discussed geometrical construction problems and their role in mathematical education. We have presented one example, worked in full details, and illustrated the most important features of construction problems, of their statements and solutions. Several examples show how a range of mathematical and computer science fields can be illustrated by construction problems or can be studied in conjunction with studying construction problems. The old construction problems can be interesting from the point of view of some quite recent theories, concepts and ideas and this shed a new light upon geometry and construction problems. In all aspects of studying construction problems it is fruitfull to use an interactive work and to exploit available software tools. WinGCLC is one such tool — a tool for teaching and studying construction problems. In future, we intend to work on further developments of the ideas presented in this paper and especially on extending WINGCLC by additional modules, some of them also for non-Euclidean geometries.

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