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Framework for Constructive Geometry

(Based on the Area Method)

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(Based on the Area Method)

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Abstract

In this technical report we present a rational reconstruction of the *area method* (developed by Chou, Gao and Zhang) for automated theorem proving for Euclidean geometry. Our rational reconstruction covers all relevant lemmas proved in full details and also full details of required algebraic reasoning (missing from the papers introducint the area method). We also present our implementation of this algorithm, made within the program GCLC.

The area method main idea is to express the hypothesis of a theorem using a set of constructive statements each of then introducing a new point, and to express the conclusion by a polynomial in a some geometry quantities, without any relation to a given system of coordinates. The proof is then developed by eliminating, in reverse order, the point introduced before, using for that purpose a set of lemmas. After eliminating all the introduced points the polynomial is just an equality between two rational expression in independent variables. Hence if they are equal the statement is true, otherwise it is false.

The proofs generated by the prover (developed as a part of GCLC) are generally short and readable. The program can prove many non-trivial theorems in a very efficient way.

keywords: automated theorem proving, Euclidian geometry, coordinate-free methods, area method.

Contents

Chapter 1

Introduction

In pursuing studies in the automation of proofs in geometry one can chose one of the two major lines of research: synthetic proof style or algebraic proof style.

Algebraic proof style has its roots in the work of Descartes[1] and in translation of geometry problems to algebraic problems. The automation of proofs along this line began with the quantifier elimination method of Tarski (30s) and since then had many developments. The characteristic set method (Wu, 2000), the elimination method (Wang, 1995), the Gröbner basis method (Kapur, 1986), and the Clifford algebra approach (Li, 2000) are examples of practical provers based on the algebraic approach. All these methods have in common the fact that the proofs do not reflect the constructive natures of the problems, are unrelated to any traditional geometric methods, and the proofs have only a yes/no conclusion.

Another approach to the automation of geometric proofs focus in synthetic proofs with the attempt to automate the traditional proof methods. Most of these methods use adding elements to the current geometric configuration so that a desired postulated will apply. The challenge is to control the explosion of the search space, because of that they use ad hoc heuristics to avoid unproductive constructions. Examples of these method include results of Gelertner (Gelernter, 1959), Nevis (Nevis, 1975), Elcock (Elcock, 1977), Greeno et. al. (Greeno *et al.*, 1979) and Coelho and Pereira (Coelho & Pereira, 1986).

In this technical report we focus on the area method, an efficient synthetic method developed by Chou, Gao, and Zhang (Chou *et al.*, 1993). This method is a base for efficient provers capable of generating human readable proofs. We present a rational reconstruction of the area method covering all relevant lemmas proved in full details and also full details of required algebraic reasoning (missing from the papers introducing the area method).

Our goal in developing a prover for this method was to extend the existing dynamic geometry tools (e.g., GCLC and Eukleides) with a module that allows formal, deductive reasoning about constructions made. For that, we need an efficient program capable of producing proofs that are human-readable, short, and with a clear justification of each proof step (e.g., lemma used, definition used etc.).

The theorem prover developed, the visualisation tools, and a database (for storing and retrieving geometric constructions and their proofs) that we developed, provide a complex framework for constructive geometry, given to all users an environment suitable for the studying an teaching geometry.

[1] Rene Descartes (1596-1650).

Chapter 2

The Area Method

We will use capital letters to denote points in the plane. We denote $\overline{AB}$ the length of the oriented segment from A to B. Thus $\overline{AB} = -\overline{BA}$.

We will denote ΔABC the triangle formed by points A, B, and C. We denote by S_{ABC} the *signed area* of the oriented triangle ΔABC (Zhang *et al.*, 1995).

Definition 1 Signed Area *The* signed area S_{ABC} *of triangle* ABC *is the usual area with a sign depending on the order of the vertices* A*,* B*, and* C*: if* $A - B - C$ *rotates counterclockwisely,* S_{ABC} *is positive, otherwise it is negative (Chou* et al.*, 1996).*

2.1 Basic Lemmas about Signed Areas and Ratio of Segments

The following properties of the signed area are used as basic lemmas (Chou *et al.*, 1996).

For any points A, B, C, and D, we have:

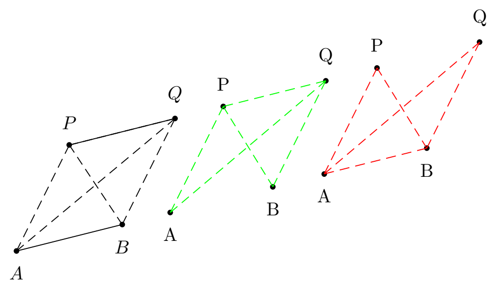
Lemma 1 $S_{ABC} = S_{CAB} = S_{BCA} = -S_{ACB} = -S_{BAC} = -S_{CBA}$.

Lemma 2 $S_{ABC} = 0$ *iff* A*,* B*, and* C *are collinear.*

Definition 2 *We use the notation* $AB \parallel PQ$ *to denote the fact that* A*,* B*,* P*, and* Q *satisfy one of the following conditions: (1)* $A = B$ *or* $P = Q$*; (2)* A*,* B*,* P*, and* Q *are on the same line; or (3) line* AB *and line* PQ *do not have a common point.*

Lemma 3 $PQ \parallel AB$ *iff* $S_{PAB} = S_{QAB}$*, i.e., iff* $S_{PAQB} = 0$.

Demonstration



Proof of $S_{PAB} = S_{QAB} \Leftrightarrow S_{PAQB} = 0$.

$S_{PAQB} = 0 \overset{\text{def}}{\Leftrightarrow} S_{PAQ} + S_{PQB} = 0 \Leftrightarrow S_{PAQ} = -S_{PQB} \Leftrightarrow S_{PAQ} = S_{QPB} \overset{(1)}{\Leftrightarrow} S_{PAB} = S_{QAB}$

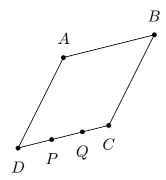
(1) $S_{PAB} = S_{QPB}$ and $S_{QAB} = S_{PAQ}$.

q.e.d.

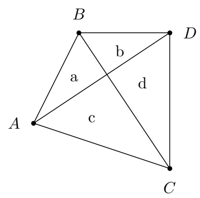
Definition 3 (Parallelogram) *A parallelogram is a quadrilateral $ABCD$ such that $AB \parallel CD$, $BC \parallel AD$, and no three vertices are on the same line.*

Definition 4 (Ratio of parallel lines) *Let $ABCD$ be a parallelogram and P, Q be two points on CD. We define the ratio of two parallel line segments as follows (Chou et al., 1993):*

$$\frac{\overline{PQ}}{\overline{AB}} = \frac{\overline{PQ}}{\overline{DC}}$$



Lemma 4 $S_{ABC} = S_{ABD} + S_{ADC} + S_{DBC}$.



$$S_{ABC} = S_{ABD} + S_{ADC} + S_{DBC} \Leftrightarrow a+c = (a+b)+(c+d)+(-b-d) \Leftrightarrow a+c = a+c$$

Lemma 5 *If points C and D are on line AB, $A \neq B$ and P is any point not on line AB (Figure 2.1) then:*

$$\frac{S_{PCD}}{S_{PAB}} = \frac{\overline{CD}}{\overline{AB}}.$$

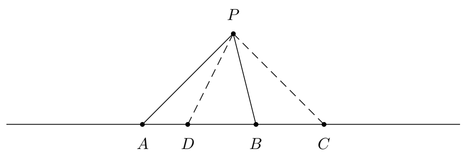


Figure 2.1: Areas, Ratios relationship

Demonstration

Given S the point in AB such that PS is perpendicular to AB ($\overline{PS}$ is the height of both triangles) then, without considering the signs

$S_{PCD} = \frac{\overline{DC} \times \overline{PS}}{2}$ and $S_{PAB} = \frac{\overline{AB} \times \overline{PS}}{2}$ so

$$\frac{S_{PCD}}{S_{PAB}} = \frac{\overline{DC} \times \overline{PS}}{2} \times \frac{2}{\overline{AB} \times \overline{PS}} = \frac{\overline{DC}}{\overline{AB}}.$$

Now looking into the signs

$$\frac{S_{PCD}}{-S_{PAB}} = \frac{-\overline{CD}}{\overline{AB}}.$$

given the fact the ΔPCD is clockwise, ΔPAB is anti-clockwise, and that $\overline{CD}$ and $\overline{AB}$ are on opposite directions.

q.e.d.

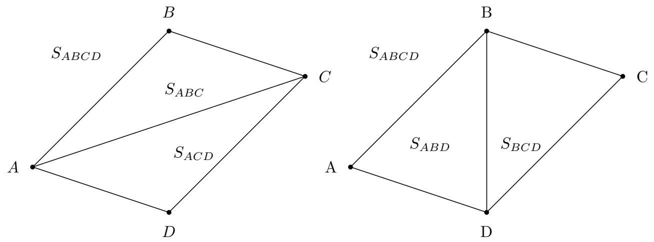
Definition 5 *S_{ABCD} is defined as $S_{ABCD} = S_{ABC}+S_{ACD}$ (Chou et al., 1996; Narboux, 2004).*

More generally we can define the signed area of an oriented n-polygon $A_1A_2 \dots A_n$, $(n \geqslant 3)$ to be (Zhang *et al.*, 1995)

$$S_{A_1A_2\dots A_n} = \sum_{i=3}^{n} S_{A_1A_{i-1}A_i}.$$

by Lemmas 1 and 4, we have.

Lemma 6 $S_{ABCD} = S_{ABC} + S_{ACD} = S_{ABD} + S_{BCD}$.



Lemma 7 $S_{ABCD} = S_{BCDA} = S_{CDAB} = S_{DABC} = -S_{ADCB} = -S_{DCBA} = -S_{CBAD} = -S_{BADC}$.

Given the fact that we are speaking about *oriented areas*, we have clock-wise, positive sign, and anticlockwise, negative sign.

Lemma 8 (EL1) *(The Co-side Theorem) Let M be the intersection of two non-parallel lines AB and PQ and $Q \neq M$ (Figure 2.2). Then:*

$$\frac{\overline{PM}}{\overline{QM}} = \frac{S_{PAB}}{S_{QAB}}; \quad \frac{\overline{PM}}{\overline{PQ}} = \frac{S_{PAB}}{S_{PAQB}}; \quad \frac{\overline{QM}}{\overline{PQ}} = \frac{S_{QAB}}{S_{PAQB}}.$$

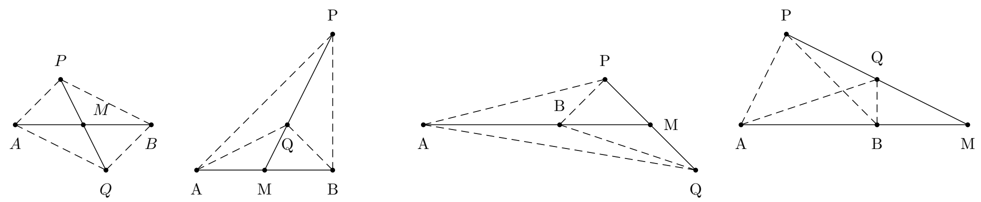
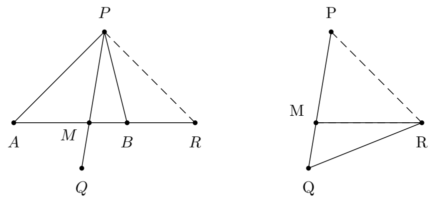


Figure 2.2: Co-side Theorem

Since P_{PAB} and S_{QAB} cannot both be zero, we always assume that the nonzero one is the denominator. Also note that $\overline{PQ} \neq 0$ since $AB \nparallel PQ$.

Demonstration Figure 2.2 gives several possible cases (in ordered geometries); the proof here presented, which is essential for unordered geometry, is valid for all cases (Zhang *et al.*, 1995). For the first formula, take a point R on AB such that $\overline{AB} = \overline{MR}$; them we have $\frac{S_{PMR}}{S_{PAB}} \overset{L5}{=} \frac{\overline{MR}}{\overline{AB}} = 1 \Leftrightarrow S_{PMR} = S_{PAB}$ the some applies for the point Q, $S_{QMR} = S_{QAB}$. So:



$$\frac{S_{PAB}}{S_{QAB}} = \frac{S_{PMR}}{S_{QMR}}$$

Now by a direct application of lemma L5 making $A = Q$, $B = D = M$, and $C = P$ we have

$$\frac{S_{PMR}}{S_{QMR}} = \frac{S_{RPM}}{S_{RQM}} \stackrel{L5}{=} \frac{\overline{PM}}{\overline{QM}}$$

in conclusion

$$\frac{S_{PAB}}{S_{QAB}} = \frac{S_{PMR}}{S_{QMR}} = \frac{\overline{PM}}{\overline{QM}}$$

The others formulas are a consequence of this first one.

q.e.d.

Lemma 9 *Let R be a point on line PQ. Then for any two points A and B (Figure 2.3) (Chou et al., 1996; Zhang et al., 1995):*

$$S_{RAB} = \frac{\overline{PR}}{\overline{PQ}} S_{QAB} + \frac{\overline{RQ}}{\overline{PQ}} S_{PAB}$$

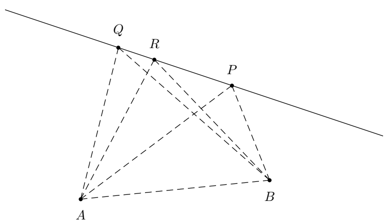
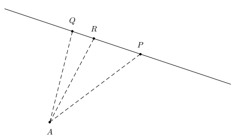


Figure 2.3:

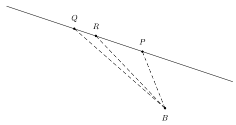
Demonstration Let $s = S_{ABPQ}$, then $S_{RAB} = s - S_{ARQ} - S_{BPR}$ (all anti-clockwise)



let $\frac{\overline{PR}}{\overline{PQ}} = \lambda$, then we have

$$\frac{S_{ARQ}}{S_{APQ}} = \frac{\overline{RQ}}{\overline{PQ}} = \frac{\overline{PQ} - \overline{PR}}{\overline{PQ}} = (1-\lambda) \Leftrightarrow S_{ARQ} = (1-\lambda)S_{APQ}$$

and,



$$\frac{S_{BPR}}{S_{BPQ}} = \frac{\overline{PR}}{\overline{PQ}} = \lambda \Leftrightarrow S_{BPR} = \lambda S_{BPQ}$$

then

$$\begin{aligned} s - S_{ARQ} - S_{BPR} &= \\ &= s - (1-\lambda)S_{APQ} - \lambda S_{BPQ} \\ &= s - (1-\lambda)(s - S_{PAB}) - \lambda(s - S_{QAB}) \\ &= s - s + \lambda s + S_{PAB} - \lambda S_{PAB} - \lambda s + \lambda S_{QAB} \\ &= \lambda S_{QAB} + (1-\lambda)S_{PAB} \end{aligned}$$

q.e.d.

For four collinear points P, Q, A, and B, such that $A \neq B$, $\frac{\overline{PQ}}{\overline{AB}}$, *the ratio of the directed segments*, is an element in $\mathbb{R}$, that satisfies the following lemmas (Chou *et al.*, 1993; Chou *et al.*, 1996; Zhang *et al.*, 1995).

Lemma 10 $\frac{\overline{PQ}}{\overline{AB}} = -\frac{\overline{QP}}{\overline{AB}} = \frac{\overline{QP}}{\overline{BA}} = -\frac{\overline{PQ}}{\overline{BA}}$.

Lemma 11 $\frac{\overline{PQ}}{\overline{AB}} = 0$ *iff* $P = Q$.

Lemma 12 $\frac{\overline{PQ}}{\overline{AB}} \times \frac{\overline{AB}}{\overline{PQ}} = 1$.

Lemma 13 $\frac{\overline{AP}}{\overline{AB}} + \frac{\overline{PB}}{\overline{AB}} = 1$.

Lemma 14 *for each* $r \in \mathbb{R}$ *there exists a unique point* P *which is collinear with* A *and* B *and satisfies* $\frac{\overline{AP}}{\overline{AB}} = r$.

Let $r = \frac{\overline{PQ}}{\overline{AB}}$. We sometimes also write $\overline{PQ} = r\overline{AB}$. A point P on line AB is determined uniquely by $\frac{\overline{AP}}{\overline{AB}}$ or $\frac{\overline{PB}}{\overline{AB}}$. We thus call

$$x_P = \frac{\overline{AP}}{\overline{AB}}, \qquad y_P = \frac{\overline{PB}}{\overline{AB}}$$

the *position ratio* or *position coordinates* of the point P with respect to AB. It is clear that $x_p + y_p = 1$.

In our machine proofs, *auxiliary parallelograms* are often added automatically and the following two propositions are used frequently.

Lemma 15 *Let $ABCD$ be a parallelogram and P be any point (fig. 2.4). Then (Chou et al., 1993; Zhang et al., 1995)*

$$S_{ABC} = S_{PAB} + S_{PCD} \qquad \text{and} \qquad S_{PAB} = S_{PDAC} = S_{PDBC}(\text{Zhang et al., 1995}).$$

$$S_{PAB} = S_{PCD} - S_{ACD} = S_{PDAC}(\text{Chou et al., 1993}).$$

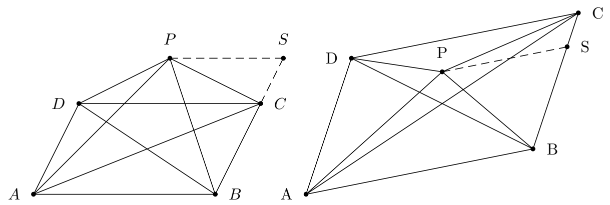


Figure 2.4:

Demonstration

Take a point S on BC such that PS is parallel to CD. By lemma 3 we have $AD \parallel BC \Leftrightarrow S_{ABC} = S_{DBC}$, $PS \parallel CD \Leftrightarrow S_{PDC} = S_{SDC} \Leftrightarrow S_{PCD} = -S_{DCS}$, and $PS \parallel AB \Leftrightarrow S_{PAB} = S_{SAB}$, $AD \parallel BS \Leftrightarrow S_{ABS} = S_{DBS} \Leftrightarrow S_{PAB} = S_{SAB} = S_{ABS} = S_{DBS} \Leftrightarrow S_{PAB} = S_{DBS}$. Therefore $S_{PAB} + S_{PDC} = S_{DBS} - S_{DCS}$. This prove the first formula. The second formula is a consequence of the first one (Zhang *et al.*, 1995).

q.e.d.

Lemma 16 *Let $ABCD$ be a parallelogram, P and Q be two points (Figure 2.5). Then (Zhang et al., 1995)*

$$S_{APQ} + S_{CPQ} = S_{BPQ} + S_{DPQ} \quad \text{or} \quad S_{PAQB} = S_{PDQC}.$$

Notice that ΔAPQ and ΔBPQ are clockwise, and ΔCPQ and ΔDPQ are anti-clockwise.

Demonstration Let O be the intersection of AC and BD. Since O is the midpoint of AC, by lemma 9, $S_{APQ} + S_{CPQ} = 2S_{OPQ}$. For the same reason, $S_{BPQ} + S_{DPQ} = 2S_{OPQ}$. We have proved the first formula, the second formula is just another form of the first one.

q.e.d.

We use a simple example to illustrate how to use these propositions to prove theorems.

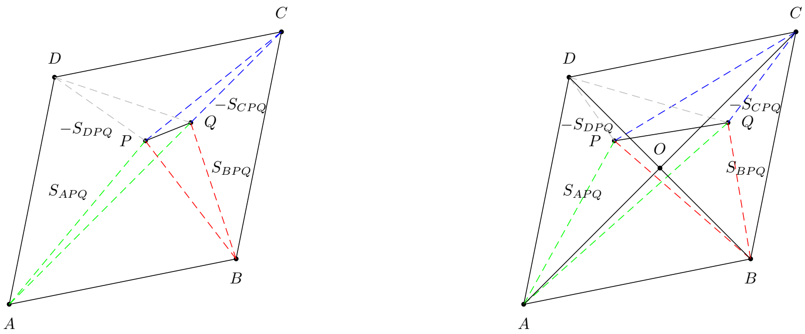


Figure 2.5:

Example 1 (Ceva's Theorem) *Let ΔABC be a triangle and P be any point in the plane. Let $D = AP \cap CB$, $E = BP \cap AC$, and $F = CP \cap AB$ (Figure 2.6). Show that:*

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1$$

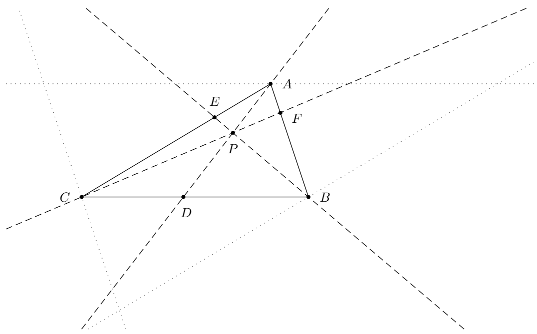


Figure 2.6:

Demonstration

Our aim is to eliminate the constructed points F, E, and D from the left hand side of the conclusion. Using the co-side theorem three times, we can eliminate E, F, and D:

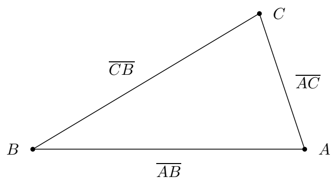
$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{S_{APC}}{S_{BCP}} \cdot \frac{S_{BPA}}{S_{CAP}} \cdot \frac{S_{CPB}}{S_{ABP}} = 1$$

q.e.d.

2.2 Basic Lemmas about Pythagoras Differences

Definition 6 (Pythagoras difference) *For three points A, B, and C, the* Pythagoras difference *P_{ABC} is defined to be:*

$$P_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2.$$



Lemma 17 $P_{AAB} = 0$.

Demonstration

$$P_{AAB} = \overline{AA}^2 + \overline{CA}^2 - \overline{AC}^2 = 0 + \overline{AC}^2 - \overline{AC}^2 = 0$$

given the fact that $\overline{CA}^2 = \overline{CA} \cdot \overline{CA} = -\overline{AC} \cdot (-\overline{AC}) = \overline{AC}^2$.

q.e.d.

Lemma 18 $P_{ABC} = P_{CBA}$.

Demonstration

$$P_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2 = \overline{CB}^2 + \overline{AB}^2 - \overline{CA}^2 = P_{CBA}$$

q.e.d.

Lemma 19 $P_{ABA} = 2\overline{AB}^2$.

Demonstration

$$P_{ABA} = \overline{AB}^2 + \overline{AB}^2 - \overline{AA}^2 = 2\overline{AB}^2$$

q.e.d.

Lemma 20 *If A, B, and C are collinear them,* $P_{ABC} = 2\overline{BA} \times \overline{BC}$.

Demonstration

$$\begin{aligned} P_{ABC} &= \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2 \\ &= \overline{AB}^2 + \overline{BC}^2 + 2\overline{AB} \times \overline{BC} - 2\overline{AB} \times \overline{BC} - \overline{AC}^2 \\ &= (\overline{AB} + \overline{BC})^2 - 2\overline{AB} \times \overline{CB} - \overline{AC}^2 \\ &= \overline{AC}^2 - \overline{AC}^2 - 2\overline{AB} \times \overline{CB} \qquad (2.1) \\ &= -2\overline{AB} \times \overline{BC} \\ &= 2\overline{BA} \times \overline{BC} \end{aligned}$$

In equation 2.1 we have $\overline{AB} + \overline{BC} = \overline{AC}$ given the fact that, by hypothesis, A, B, and C are collinear.

q.e.d.

Definition 7 *For a quadrilateral $ABCD$, we have*

$$P_{ABCD} = P_{ABD} - P_{CBD} = \overline{AB}^2 + \overline{CD}^2 - \overline{BC}^2 - \overline{DA}^2.$$

Lemma 21 *We have $P_{ABCD} = -P_{ADCB} = P_{BADC} = -P_{BCDA} = P_{CDAB} = -P_{CBAD} = P_{DCBA} = -P_{DABC}$.*

Demonstration

$$\begin{aligned}
P_{ADCB} &= \overline{AD}^2 + \overline{CB}^2 - \overline{DC}^2 - \overline{BA}^2 \\
&= -\overline{AB}^2 - \overline{CD}^2 + \overline{BC}^2 + \overline{DA}^2 \\
&= -(P_{ABCD}) && (2.2) \\
P_{BADC} &= \overline{BA}^2 + \overline{DC}^2 - \overline{AD}^2 - \overline{CB}^2 \\
&= \overline{AB}^2 + \overline{CD}^2 - \overline{BC}^2 - \overline{DA}^2 \\
&= P_{ABCD} && (2.3) \\
P_{BCDA} &= \overline{BC}^2 + \overline{DA}^2 - \overline{CD}^2 - \overline{AB}^2 \\
&= -\overline{AB}^2 - \overline{CD}^2 + \overline{BC}^2 + \overline{DA}^2 \\
&= -(P_{ABCD}) && (2.4) \\
P_{CDAB} &= \overline{CD}^2 + \overline{AB}^2 - \overline{DA}^2 - \overline{BC}^2 \\
&= \overline{AB}^2 + \overline{CD}^2 - \overline{BC}^2 - \overline{DA}^2 \\
&= P_{ABCD} && (2.5) \\
P_{CBAD} &= \overline{CB}^2 + \overline{AD}^2 - \overline{BA}^2 - \overline{DC}^2 \\
&= -\overline{AB}^2 - \overline{CD}^2 + \overline{BC}^2 + \overline{DA}^2 \\
&= -(P_{ABCD}) && (2.6) \\
P_{DCBA} &= \overline{DC}^2 + \overline{BA}^2 - \overline{CB}^2 - \overline{AD}^2 \\
&= \overline{AB}^2 + \overline{CD}^2 - \overline{BC}^2 - \overline{DA}^2 \\
&= P_{ABCD} && (2.7) \\
P_{DABC} &= \overline{DA}^2 + \overline{BC}^2 - \overline{AB}^2 - \overline{CD}^2 \\
&= -\overline{AB}^2 - \overline{CD}^2 + \overline{BC}^2 + \overline{DA}^2 \\
&= -(P_{ABCD}) && (2.8)
\end{aligned}$$

q.e.d.

Definition 8 *For four points A, B, C, and D, the notation $AB \perp CD$ implies that one of the following conditions is true: $A = B$, or $C = D$, or the line AB is perpendicular to line CD.*

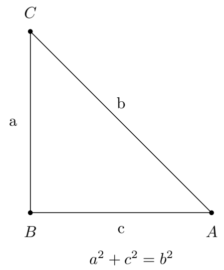
Lemma 22 (Pythagorean Theorem) *$AB \perp BC$ iff $P_{ABC} = 0$.*

Demonstration

If $A = B$, we have $A = B = C$, D, then $\overline{AB}^2 = \overline{CB}^2 = \overline{AC}^2 = 0$.

If $C = D$, we have A, $B = C = D$, then $\overline{AB}^2 + \overline{CB}^2 - \overline{AB}^2 = \overline{AC}^2 + 0 - \overline{AC}^2 = 0$.

If $A \neq B$ and $B \neq C$ then we have a right triangle;



e.g. $\overline{AB}^2 + \overline{BC}^2 = \overline{CA}^2$.

q.e.d.

Lemma 23 *$AB \perp CD$ iff $P_{ACD} = P_{BCD}$ or $P_{ACBD} = 0$ (figure 2.7).*

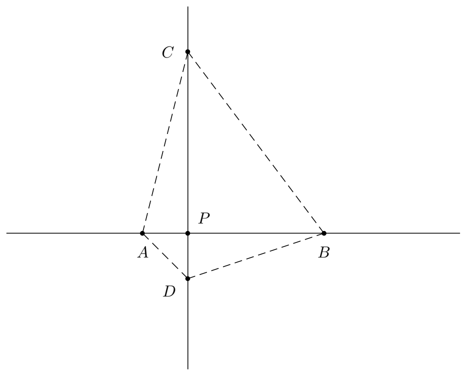


Figure 2.7:

Demonstration

Let P be the intersection of lines AB and CD, then:

$$\begin{aligned}
\overline{AD}^2 &= \overline{AP}^2 + \overline{PD}^2 \\
\overline{AC}^2 &= \overline{AP}^2 + \overline{PC}^2 \\
\overline{AD}^2 - \overline{PD}^2 &= \overline{AC}^2 - \overline{PC}^2 \\
\overline{BD}^2 &= \overline{BP}^2 + \overline{PD}^2 \\
\overline{BC}^2 &= \overline{BP}^2 + \overline{PC}^2 \\
\overline{BD}^2 - \overline{PD}^2 &= \overline{BC}^2 - \overline{PC}^2 \\
\overline{AD}^2 - \overline{AC}^2 &= \overline{PD}^2 + \overline{PC}^2 \\
\overline{BD}^2 - \overline{BC}^2 &= \overline{PD}^2 + \overline{PC}^2 \\
\overline{AD}^2 - \overline{AC}^2 &= \overline{BD}^2 - \overline{BC}^2 \\
\overline{AC}^2 - \overline{AD}^2 &= \overline{BC}^2 - \overline{BD}^2 \\
\overline{AC}^2 + \overline{DC}^2 - \overline{AD}^2 &= \overline{BC}^2 + \overline{DC}^2 - \overline{BD}^2 \\
P_{ACD} &= P_{BCD}
\end{aligned}$$

q.e.d.

The above generalised Pythagorean proposition is one of the most useful tools in our mechanical theorem proving method.

Lemma 24 *Let D be the foot of the perpendicular drawn from point P to a line AB (figure 2.8). Then:*

$$\frac{\overline{AD}}{\overline{DB}} = \frac{P_{PAB}}{P_{PBA}}, \quad \frac{\overline{AD}}{\overline{AB}} = \frac{P_{PAB}}{2\overline{AB}^2}, \quad \frac{\overline{DB}}{\overline{AB}} = \frac{P_{PBA}}{2\overline{AB}^2}.$$

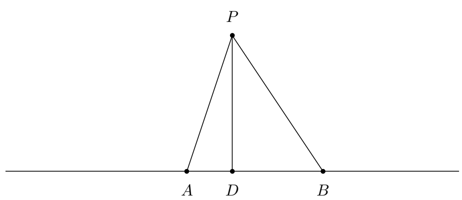


Figure 2.8:

Lemma 25 *Let AB and PQ be two non-perpendicular lines, and Y be the intersection of line PQ and the line passing through A and perpendicular to AB (figure 2.9). Then:*

$$\frac{\overline{PY}}{\overline{QY}} = \frac{P_{PAB}}{P_{QAB}}, \quad \frac{\overline{PY}}{\overline{PQ}} = \frac{P_{PAB}}{P_{PAQB}}, \quad \frac{\overline{QY}}{\overline{PQ}} = \frac{P_{QAB}}{P_{PAQB}}.$$

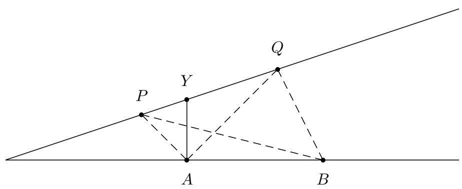


Figure 2.9:

Lemma 26 *Let R be a point on line PQ with position ratios $r_1 = \frac{\overline{PR}}{\overline{PQ}}$, $r_2 = \frac{\overline{RQ}}{\overline{PQ}}$ with respect to PQ. Then for points A, B, we have (figure 2.10):*

$$\begin{aligned} P_{RAB} &= r_1 P_{QAB} + r_2 P_{PAB} \\ P_{ARB} &= r_1 P_{AQB} + r_2 P_{APB} - r_1 r_2 P_{PQP}\ . \end{aligned}$$

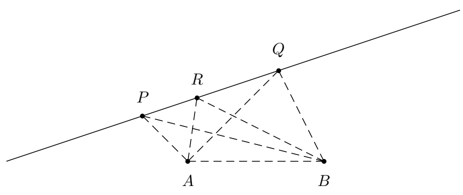


Figure 2.10:

Lemma 27 *Let $ABCD$ be a parallelogram. Then for any points P and Q, we have (figure 2.11):*

$$\begin{aligned} P_{APQ} + P_{CPQ} &= P_{BPQ} + P_{DPQ} \quad \text{or} \quad P_{APBQ} = P_{DPCQ} \\ P_{PAQ} + P_{PCQ} &= P_{PBQ} + P_{PDQ} + 2P_{BAD}\ . \end{aligned}$$

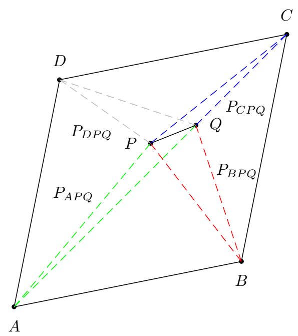


Figure 2.11:

2.3 The Constructive Geometry Statements

2.3.1 Constructive Geometry Statements

Points are the basic geometry objects, from which we can introduce two other *geometric objects*: lines and circles.

A *straight line* can be given in one of the following four forms.

(LINE U V) is the line passing through two points U and V.

(PLINE W U V) is the line passing through point W and parallel to (LINE U V).

(TLINE W U V) is the line passing through point W and perpendicular to (LINE U V).

(BLINE U V) is the perpendicular bisector of UV.

To make sure that the four kinds of lines are well defined, we need to assume that $U \neq V$ which is called the *non-degenerated condition* (ndg) of the corresponding line.

(CIR O U) denotes a *circle* with point O as its centre and passing through point U.

Constructions: a *construction* is one of the following ways of introducing new points. For each construction, we also give its ndg condition and the degrees of freedom for the constructed point.

C1 – (POINT[S] Y1, . . . , Yn). Take arbitrary points $Y_1, \ldots, Y_n$ in the plane. Each Y_i has two degrees of freedom.

C2 – (ON Y ln). Take a point Y on a line ln. The ndg condition of C2 is the ndg condition of the line ln. A point Y has one degree of freedom.

C3 – (ON Y (Cir O P)). Take a point Y on a circle (CIR O P). The ndg condition is $O \neq P$. Point Y has one degree of freedom.

C4 – (INTER Y ln1 ln2). Point Y is the intersection of line ln_1 and line ln_2. Point Y is a fixed point. The ndg condition is $ln_1 \nparallel ln_2$. More precisely, we have:

1. If ln_1 is (LINE U V) or (PLINE W U V) and ln_2 is (LINE P Q) or (PLINE R P Q), the ndg condition is $UV \nparallel PQ$.
2. If ln_1 is (LINE U V) or (PLINE W U V) and ln_2 is (BLINE P Q) or (TLINE R P Q), then the ndg condition is $UV \not\perp PQ$.
3. If ln_1 is (BLINE U V) or (TLINE W U V) and ln_2 is (BLINE P Q) or (TLINE R P Q), then the ndg condition is $UV \nparallel PQ$.

C5 – (INTER Y ln (CIR O P)). Point Y is the intersection of line ln and circle (CIR O P) other than point P. Line ln could be (LINE P U), (PLINE P U V), or (TLINE P U V). The ndg conditions are $O \neq P$, $Y \neq P$, and line ln is not degenerate. Point Y is a fixed point.

C6 – (INTER Y (CIR O1 P) (CIR O2 P)). Point Y is the intersection of the circle (CIR O1 P) and the circle (CIR O2 P) other them point P. The ndg condition is that O_1, O_2, and P are not collinear. Point Y is a fixed point.

C7 – (PRATIO Y W U V r). Take a point Y on the line (PLINE W U V) such that $\overline{WY} = r\overline{UV}$, where r can be a rational number, a rational expression in geometric quantities, or a variable.

If r is a fixed quantity the Y is a fixed point; if r is a variable the Y has one degree of freedom. The ndg condition is $U \neq V$. If r is a rational expression in geometric quantities the we will further assume that the denominator of r could not be zero.

C8 – (TRATIO Y U V r). Take a point Y on line (TLINE U U V) such that $r = \frac{4S_{UVY}}{P_{UVU}} (= \frac{\overline{UY}}{\overline{UV}})$, where r can be a rational number, a rational expression in geometric quantities, or a variable.

If r is a fixed quantity then Y is a fixed point; if r is a variable Y has one degree of freedom. The ndg condition is the same as that of C7.

Since there are four kinds of lines, constructions C2, C4, C5 have 4, 10, 3 possible forms respectively. Thus, totally we have 22 different forms of constructions.

Definition 9 (Class of Constructive Geometry Statements) *The class of* Constructive Geometry Statements*,* **C***, is the class of statements defined as follows. A statement in class* **C** *is a list* $S = (C_1, C_2, \ldots, C_n, G)$ *where* C_i *for* $1 \leqslant i \leqslant n$ *are constructions such that each* C_i *introduces a new point from the points introduced before; and* $G = (E_1, E_2)$ *where* E_1 *and* E_2 *are polynomials in geometric quantities of the points introduced by the* C_i *and* $E_1 = E_2$ *is the conclusion of the statement.*

Let $S = (C_1, C_2, \ldots, C_n, (E_1, E_2))$ be a statement in **C**. The ndg condition of S is the set of ndg conditions of the C_is plus the condition that the denominators of the length ratios in E_1 and E_2 are not equal to zero.

The 22 constructions are not independent to each other. We now introduce a minimal set of constructions which are equivalent to all the 22 constructions but much few in number.

A *minimal set of constructions* consist of C1, C7, C8, and the following two constructions.

C41 – (INTER Y (LINE U V) (LINE P Q)) .

C42 – (FOOT Y P U V) or equivalently (INTER Y (LINE U V) (TLINE P U V)). The ndg condition is $U \neq V$.

We first show how to represent the four kinds of lines by one kind: (LINE U V).

For ln =(PLINE W U V), we first introduce a new point N by (PRATIO N W U V 1). Then ln =(LINE W N).

For ln =(TLINE W U V), we have two cases: if W, U, V are collinear, ln =(LINE N W) where N is introduced by (TRATIO N W U 1); otherwise ln =(LINE N W) where N is given by (FOOT N W U V).

(BLINE U V) can be written as (LINE N M) where N and M are introduced as follows (MIDPOINT M U V) (i.e. (PRATIO M U U V 1/2)), and (TRATIO N M U 1).

Since now there is only one kind of line, to represent all the 22 constructions by the constructions in the minimal set we only need to consider the following cases.

- (ON Y (LINE U V)) is equivalent to (PRATIO Y U U V r) where r is an *indeterminate.*
- (INTER Y (LINE U V) (CIR O U)) is equivalent to two constructions: (FOOT N O U V), (PRATIO Y N N U -1).
- C6 can be reduced to (FOOT N P O1 O2) and (PRATIO Y N N P -1).
- For C3, i.e., to take an arbitrary point Y on a circle (CIR O P), we first take an arbitrary point Q. Then Y is introduced by (INTER Y (LINE P Q) (CIR O P)).

2.4 The Algorithm

The key step of the method is to *eliminate points from geometry quantities*. The point are introduced naturally and are eliminated from the conclusion in the reverse order.

2.4.1 The Elimination Procedures

Considering only the minimal set of constructions: C1, C7, C8, C41, and C42 we need only to eliminate points introduced by four constructions from three kinds of geometry quantities.

Lemma 28 *Let $G(Y)$ be one of the following geometry quantities: S_{ABY}, S_{ABCY}, P_{ABY}, or P_{ABCY} for distinct points A, B, C, and Y. For three collinear points Y, U, and V, by lemmas 9 and 26 we have:*

$$G(Y) = \frac{\overline{UY}}{\overline{UV}} G(V) + \frac{\overline{YV}}{\overline{UV}} G(U). \tag{2.9}$$

We call $G(Y)$ a *linear geometry quantity* for variable Y. Elimination procedures for all linear geometry quantities are similar for constructions C7, C41, and C42.

Demonstration

$$\begin{aligned}
G(Y) &= S_{ABY} \\
S_{ABY} &= S_{YAB} \quad \text{by lemma 1} \\
&= \frac{\overline{UY}}{\overline{UV}} S_{VAB} + \frac{\overline{YV}}{\overline{UV}} S_{UAB} \quad \text{by lemma 9; } U,\ V, \text{ and } Y \text{ are collinear} \\
&= \frac{\overline{UY}}{\overline{UV}} S_{ABV} + \frac{\overline{YV}}{\overline{UV}} S_{ABU} \quad \text{by lemma 1} \\
&= \frac{\overline{UY}}{\overline{UV}} G(V) + \frac{\overline{YV}}{\overline{UV}} G(U)
\end{aligned}$$

$$\begin{aligned}
G(Y) &= P_{ABY} \\
P_{ABY} &= P_{YBA} \quad \text{by lemma 18} \\
&= \frac{\overline{UY}}{\overline{UV}} P_{VBA} + \frac{\overline{YV}}{\overline{UV}} P_{UBA} \quad \text{by lemma 26; } U,\ V, \text{ and } Y \text{ are collinear} \\
&= \frac{\overline{UY}}{\overline{UV}} P_{ABV} + \frac{\overline{YV}}{\overline{UV}} P_{ABU} \quad \text{by lemma 18} \\
&= \frac{\overline{UY}}{\overline{UV}} G(V) + \frac{\overline{YV}}{\overline{UV}} G(U)
\end{aligned}$$

$$\begin{aligned}
G(Y) &= S_{ABCY} \overset{def\ 5}{=} S_{ABC} - P_{ACY} \\
&= S_{ABC} + \frac{\overline{UY}}{\overline{UV}} S_{ABC} - \frac{\overline{UY}}{\overline{UV}} S_{ABC} + \frac{\overline{YV}}{\overline{UV}} S_{ABC} - \frac{\overline{YV}}{\overline{UV}} S_{ABC} + S_{ACY}
\end{aligned}$$

$$
\begin{aligned}
&= (1-(\frac{\overline{UY}}{\overline{UV}}+\frac{\overline{YV}}{\overline{UV}})S_{ABC}+\frac{\overline{UY}}{\overline{UV}}S_{ABC}+\frac{\overline{YV}}{\overline{UV}}S_{ABC}+S_{ACY}\\
&= 0+\frac{\overline{UY}}{\overline{UV}}S_{ABC}+\frac{\overline{YV}}{\overline{UV}}S_{ABC}+S_{ACY} \quad U,\ V, \text{ and } Y \text{ are collinear}\\
&= \frac{\overline{UY}}{\overline{UV}}S_{ABC}+\frac{\overline{YV}}{\overline{UV}}S_{ABC}+S_{YAC} \quad \text{by lemma 1}\\
&= \frac{\overline{UY}}{\overline{UV}}S_{ABC}+\frac{\overline{UY}}{\overline{UV}}S_{ACV}+\frac{\overline{YV}}{\overline{UV}}S_{ABC}+\frac{\overline{YV}}{\overline{UV}}S_{ACU} \quad \text{by lemma 9; } U,\ V, \text{ and } Y \text{ are collinear}\\
&= \frac{\overline{UY}}{\overline{UV}}S_{ABCV}+\frac{\overline{YV}}{\overline{UV}}S_{ABCU} \quad \text{by definition 5}\\
&= \frac{\overline{UY}}{\overline{UV}}G(V)+\frac{\overline{YV}}{\overline{UV}}G(U)
\end{aligned}
$$

$$
\begin{aligned}
G(Y) &= P_{ABCY} \overset{def\ 7}{=} P_{ABY}-P_{CBY}\\
&= \frac{\overline{UY}}{\overline{UV}}P_{ABV}+\frac{\overline{YV}}{\overline{UV}}P_{ABU}-(\frac{\overline{UY}}{\overline{UV}}P_{CBV}+\frac{\overline{YV}}{\overline{UV}}P_{CBU})\\
&= \frac{\overline{UY}}{\overline{UV}}(P_{ABV}-P_{CBV})+\frac{\overline{YV}}{\overline{UV}}(P_{ABU}-P_{CBU})\\
&= \frac{\overline{UY}}{\overline{UV}}P_{ABCV}+\frac{\overline{YV}}{\overline{UV}}P_{ABCU} \quad \text{by definition 7}\\
&= \frac{\overline{UY}}{\overline{UV}}G(V)+\frac{\overline{YV}}{\overline{UV}}G(U)
\end{aligned}
$$

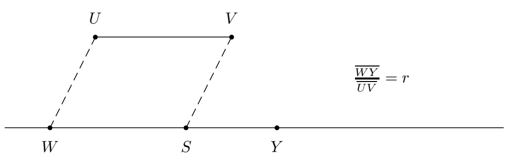
q.e.d.

Lemma 29 (EL2) *Let $G(Y)$ be a linear geometry quantity and point Y be introduced by construction (PRATIO Y W U V r). Then we have:*

$$G(Y) = G(W) + r(G(V) - G(U)).$$

Demonstration

Take a point S such that $\overline{WS} = \overline{UV}$.



By (2.9)[U:=A,V:=B,W;=U,S:=V]

$$\begin{aligned} G(Y) &= \frac{\overline{WY}}{\overline{WS}}G(S) + \frac{\overline{YS}}{\overline{WS}}G(W) & \frac{\overline{WY}}{\overline{WS}} = 1, \text{by hypothesis} \\ &= rG(S) + \left(\frac{\overline{WY} - \overline{WS}}{\overline{WS}}\right)G(W) & W,\ Y,\ S \text{ are collinear} \\ &= rG(S) + (1-r)G(W) \end{aligned}$$

By lemmas 16 ($S_{APQ} = S_{BPQ} + S_{DPQ} - S_{CPQ}$) and 27, ($P_{APQ} = P_{BPQ} + P_{DPQ} - +P_{CPQ}$) considering the parallelogram $UVSW$ and the points W and Y we have $G(S) = G(W) + G(V) - G(U)$. Substituting this into the above equation, we obtain the result.

$$\begin{aligned} G(Y) &= rG(S) + (1-r)G(W) \\ &= r(G(W) + G(V) - G(U)) + (1-r)G(W) \\ &= rG(W) - rG(W) + G(W) + r(G(V) - G(U)) \\ &= G(W) + r(G(V) - G(U)) \end{aligned}$$

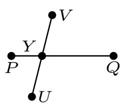
Notice that we need the ndg condition $U \neq V$.

q.e.d.

Lemma 30 (EL3) *Let $G(Y)$ be a linear geometry quantity and point Y be introduced by construction (INTER Y (LINE U V) (LINE P Q). Then we have:*

$$G(Y) = \frac{S_{UPQ}G(V) - S_{VPQ}G(U)}{S_{UPVQ}}.$$

Demonstration



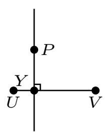
By the co-side theorem[P:=U,Q:=V,A:=P,B:=Q,M:=Y], $\frac{\overline{UY}}{\overline{UV}} = \frac{S_{UPQ}}{S_{UPVQ}}$, $\frac{\overline{YV}}{\overline{UV}} = -\frac{\overline{VY}}{\overline{UV}} - \frac{S_{VPQ}}{S_{UPVQ}}$. Substituting these into equation (2.9), we prove the result.

q.e.d.

Lemma 31 (EL4) *Let $G(Y)$ be a linear geometry quantity ($\neq P_{AYB}$) and point Y be introduced by construction (FOOT Y P U V). Then we have:*

$$G(Y) = \frac{P_{PUV}G(V) + P_{PVU}G(U)}{2\overline{UV}^2}.$$

Demonstration



By lemma 24[A:=U,B:=V,D:=Y], $\frac{\overline{UY}}{\overline{UV}} = \frac{P_{PUV}}{2\overline{UV}^2}$, $\frac{\overline{YV}}{\overline{UV}} = \frac{P_{PVU}}{2\overline{UV}^2}$. Substituting these into (2.9), we prove the result.

q.e.d.

Lemma 32 (EL5) *Let $G(Y) = P_{AYB}$ and point Y be introduced by construction (FOOT Y P U V) or (INTER Y (LINE U V) (LINE P Q)). Then we have:*

$$G(Y) = \frac{\overline{UY}}{\overline{UV}}G(V) + \frac{\overline{YV}}{\overline{UV}}G(U) - \frac{\overline{UY}}{\overline{UV}} \cdot \frac{\overline{YV}}{\overline{UV}}P_{UVU}. \tag{2.10}$$

Demonstration

By lemma 26[R:=Y,P:=U,Q:=V], for three collinear points Y, U, and V, we have $r_1 = \frac{\overline{UY}}{\overline{UV}}$, $r_2 = \frac{\overline{YV}}{\overline{UV}}$, and $P_{AYB} = r_1 P_{AVB} + r_2 P_{AUB} - r_1 r_2 P_{UVU}$, that is, $G(Y) = \frac{\overline{UY}}{\overline{UV}}G(V) + \frac{\overline{YV}}{\overline{UV}}G(U) - \frac{\overline{UY}}{\overline{UV}} \cdot \frac{\overline{YV}}{\overline{UV}}P_{UVU}$.

q.e.d.

Lemma 33 (EL6) *Let Y be introduced by (PRATIO Y W U V r). Then we have:*

$$P_{AYB} = P_{AWB} + r(P_{AVB} - P_{AUB} + P_{WUV}) - r(1-r)P_{UVU}.$$

Construction C8 needs special treatment.

Lemma 34 (EL7) *Let Y be introduced by (TRATIO Y P Q r). Then we have:*

$$S_{ABY} = S_{ABP} - \frac{r}{4}P_{PAQB}.$$

Demonstration

Let A_1 be the orthogonal projection from A to PQ. Then by lemmas 3 and 24:

$$\frac{S_{PAY}}{S_{PQY}} = \frac{S_{PA_1Y}}{S_{PQY}} = \frac{\overline{PA_1}}{\overline{PQ}} = \frac{P_{A_1PQ}}{P_{QPQ}} = \frac{P_{APQ}}{P_{QPQ}}$$

Thus $S_{PAY} = \frac{P_{APQ}}{P_{QPQ}}S_{PQY} = \frac{r}{4}P_{APQ}$. Similarly, $S_{PBY} = \frac{P_{BPQ}}{P_{QPQ}}S_{PQY} = \frac{r}{4}P_{BPQ}$. Now $S_{ABY} = S_{ABP} + S_{PBY} - S_{PAY} = S_{ABP} - \frac{r}{4}P_{PAQB}$.

q.e.d.

Lemma 35 (EL8) *Let Y be introduced by (TRATIO Y P Q r). Then we have:*

$$P_{ABY} = P_{ABP} - 4rS_{PAQB}.$$

Demonstration

Let the orthogonal projections from A and B to PY be A_1 and B_1. Then

$$\frac{P_{BPAY}}{P_{YPY}} = \frac{P_{B_1PA_1Y}}{P_{YPY}} = \frac{\overline{A_1B_1}}{\overline{PY}} = \frac{S_{PA_1QB_1}}{S_{PQY}} = \frac{S_{PAQB}}{S_{PQY}}.$$

Since $PY \perp PQ$, $S_{PQY}^2 = \frac{1}{4}\overline{PQ}^2 \cdot \overline{PY}^2$. Then $P_{YPY} = 2\overline{PY}^2 = 4rS_{PQY}$. Therefore $P_{ABY} = P_{ABP} - P_{BPAY} = P_{ABP} - 4rS_{PAQB}$.

q.e.d.

Lemma 36 (EL9) *Let Y be introduced by (TRATIO Y P Q r). Then we have:*

$$P_{AYB} = P_{APB} + r^2 P_{PQP} - 4r(S_{APQ} + S_{BPQ}).$$

Demonstration

By Lemma 35

$$P_{APY} = 4rS_{APQ}, \quad P_{BPY} = 4rS_{BPQ}.$$

Then

$$P_{YPY} = 2\overline{PY}^2 = 4rS_{PQY} = r^2 P_{PQP}$$

Then

$$P_{AYB} = P_{APB} - P_{APY} - P_{BPY} + P_{YPY} = P_{APB} + r^2 P_{PQP} - 4r(S_{APQ} + S_{BPQ}).$$

q.e.d.

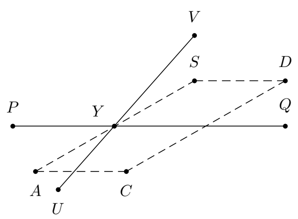
Now we consider how to eliminate points from the ratio of lengths.

Lemma 37 (EL10) *Let Y be introduced by (INTER Y (LINE U V) (LINE P Q)). The we have:*

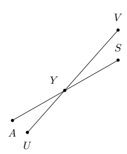
$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{S_{AUV}}{S_{CUDV}} & \text{if } A \text{ is not on } UV \\ \frac{S_{APQ}}{S_{CPDQ}} & \text{otherwise} \end{cases}$$

Demonstration

If A is not on UV, let S be a point such that $\overline{AS} = \overline{CD}$.

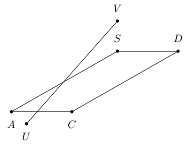


$$\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AY}}{\overline{AS}} \quad \text{by construction}$$



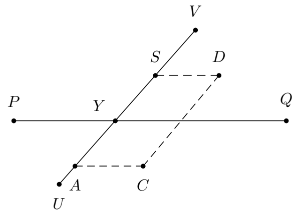
$$= \frac{S_{AUV}}{S_{AUSV}} \quad \text{by lemma 8}$$

$$= \frac{S_{AUV}}{S_{UAVS}} \quad \text{by lemma 21}$$



$$\begin{aligned} &= \frac{S_{AUV}}{S_{UCVD}} && \text{by lemma 16} \\ &= \frac{S_{AUV}}{S_{CUDV}} && \text{by lemma 21} \end{aligned}$$

If A is on UV



$$\begin{aligned} \frac{\overline{AY}}{\overline{CD}} &= \frac{\overline{AY}}{\overline{AS}} && \text{by construction} \\ &= \frac{S_{APQ}}{S_{APSQ}} && \text{by lemma 8} \\ &= \frac{S_{APQ}}{S_{CPDQ}} \end{aligned}$$

q.e.d.

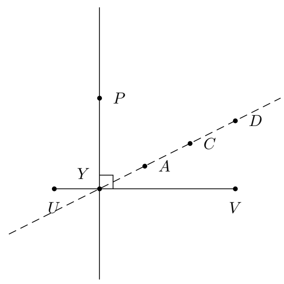
Lemma 38 (EL11) *Let Y be introduced by (FOOT Y P U V). We assume $D \neq U$; otherwise interchange U and V.*

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{P_{PCAD}}{P_{CDC}} & \textit{if } A \textit{ is on } UV \\ \frac{S_{AUV}}{S_{CUDV}} & \textit{otherwise} \end{cases}$$

Demonstration

If A is on UV, let T be a point such that $\overline{AT} = \overline{CD}$. By lemma 24 and 27, $\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AY}}{\overline{AT}} = \frac{P_{PAT}}{P_{ATA}} = \frac{P_{PCAD}}{P_{CDC}}$.

The second equation is a direct consequence of the co-side theorem.



By the co-side theorem (lemma 8) with line CD and UV we have:

$$\frac{\overline{CY}}{\overline{CD}} = \frac{S_{CUV}}{S_{CUDV}}$$

and also by the co-side theorem with line AC and UV we have:

$$\frac{\overline{CY}}{\overline{AY}} = \frac{S_{CUV}}{S_{AUV}} \Leftrightarrow \overline{AY} = \frac{\overline{CY} S_{AUV}}{S_{CUV}}$$

so:

$$\frac{\overline{AY}}{\overline{CD}} = \frac{\frac{\overline{CY} S_{AUV}}{S_{CUV}}}{\overline{CD}} = \frac{\overline{CY}}{\overline{CD}} \cdot \frac{S_{AUV}}{S_{CUV}} = \frac{S_{CUV}}{S_{CUDV}} \cdot \frac{S_{AUV}}{S_{CUV}} = \frac{S_{AUV}}{S_{CUDV}}$$

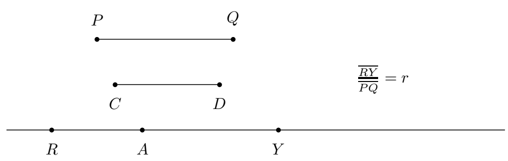
q.e.d.

Lemma 39 (EL12) *Let Y be introduced by (PRATIO Y R P Q r). Then we have*

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\frac{\overline{AR}}{\overline{PQ}}+r}{\frac{\overline{CD}}{\overline{PQ}}} & \textit{if } A \textit{ is on } RY \\ \frac{S_{APRQ}}{S_{CPDQ}} & \textit{otherwise} \end{cases}$$

Demonstration

The first case is obvious:



$$\frac{\overline{AY}}{\overline{CD}} = \frac{\frac{\overline{AY}}{\overline{PQ}}}{\frac{\overline{CD}}{\overline{PQ}}} = \frac{\frac{\overline{AR}+\overline{AY}}{\overline{PQ}}}{\frac{\overline{CD}}{\overline{PQ}}} = \frac{\frac{\overline{AR}}{\overline{PQ}}+r}{\frac{\overline{CD}}{\overline{PQ}}}$$

The second case, take points T and S such that $\frac{\overline{RT}}{\overline{PQ}} = 1$ and $\frac{\overline{AS}}{\overline{CD}} = 1$. By the co-side theorem, $\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AY}}{\overline{AS}} = \frac{S_{ART}}{S_{ARST}} = \frac{S_{APRQ}}{S_{CPDQ}}$.

q.e.d.

Lemma 40 (EL13) *Let Y be introduced by (TRATIO Y P Q r). Then we have*

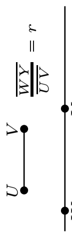
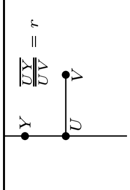
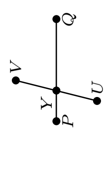
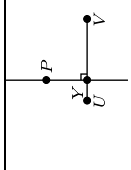
$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{P_{APQ}}{P_{CPDQ}} & \textit{if } A \textit{ is on } PY \\ \frac{S_{APQ}-\frac{r}{4}P_{PQP}}{S_{CPDQ}} & \textit{otherwise} \end{cases}$$

Demonstration

The first case is a direct consequence of lemma 25. If A is on PY, then $\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AP}}{\overline{CD}} - \frac{\overline{YP}}{\overline{CD}}$. By the co-side theorem, $\frac{\overline{AP}}{\overline{CD}} = \frac{S_{APQ}}{S_{CPDQ}}$; $\frac{\overline{AY}}{\overline{CD}} = \frac{S_{YPQ}}{S_{CPDQ}} = \frac{rP_{PQP}}{4S_{CPDQ}}$. Now the second result follows immediately

q.e.d.

2.4.2 Table of Elimination Lemmas

		Construction	ndc	Elimination formulas						
				P_{AYB}	P_{ABY}	P_{ABCY}	S_{ABY}	S_{ABCY}	$\frac{\overline{AY}}{\overline{CD}}$	$\frac{\overline{AY}}{\overline{BY}}$
1	C7		$U \neq V$, if $r = \frac{r1}{r2}$ then $r2 \neq 0$	EL6	EL2				EL12	
2	C8		$U \neq V$, if $r = \frac{r1}{r2}$ then $r2 \neq 0$	EL9	EL8		EL7		EL13	
3	C41			EL5	EL3				EL10	EL1
4	C42		$U \neq V$	EL5	EL4				EL11	
				A	B		C		D	E

C7 – (PRATIO Y W U V r)
C8 – (TRATIO Y U V r)
C41 – (INTER Y (LINE U V) (LINE P Q))
C42 – (FOOT Y P U V)

2.4.3 Free Points and the Algorithm

For a geometry statement $S = (C_1, C_2, \ldots, C_m, (E, F))$, after eliminating all the non-free points introduced by C_i from E and F using the lemmas in the preceding subsections, we obtain two rational expression E' and F' in indeterminates, areas and Pythagoras differences of *free points*. These geometric quantities are generally not independent, e.g. for any four points A, B, C, and D we have

$$S_{ABC} = S_{ABD} + S_{ADC} + S_{DBC}$$

We thus need to reduce E' and F' to expressions in independent variables. To do that, we need the concept of area coordinates.

Definition 10 (Area Coordinates) *Let A, O, U, and V be four points such that O, U, and V are not collinear. The area coordinates of A with respect to OUV are*

$$x_A = \frac{S_{OUA}}{S_{OUV}}, \quad y_A = \frac{S_{OAV}}{S_{OUV}}, \quad z_A = \frac{S_{AUV}}{S_{OUV}}.$$

It is clear that $x_A + y_A + z_A = 1$. Since x_A, y_A, and z_A are not independent, we also call x_A, y_A the area coordinates of A with respect to OUV.

It is clear that the points in the plane are in a one to one correspondence with their area coordinates. To represent E and F as expressions in independent variables, we first introduce three new points O, U, and V, such that, $UO \perp OV$. We will reduce E and F to expressions in the area coordinates of the free points with respect to OUV.

For any free points A, B, and C, we have the following results.

Lemma 41 $S_{ABC} = \frac{(S_{OVB}-S_{OVC})S_{OUA}+(S_{OVC}-S_{OVA})S_{OUB}+(S_{OVA}-S_{OVB})S_{OUC}}{S_{OUV}}$.

Lemma 42 $P_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AB}^2$

Lemma 43 $\overline{AB}^2 = \frac{\overline{OU}^2(S_{OVA}-S_{OVB})^2}{S^2_{OUV}} + \frac{\overline{OV}^2(S_{OUA}-S_{OUB})^2}{S^2_{OUV}}$.

Lemma 44 $S^2_{OUV} = \frac{\overline{OU}^2 \cdot \overline{OV}^2}{4}$.

Demonstration For the proof of 41, see Case 15 of algorithm ELIM in (Zhang *et al.*, 1995) (or Zhang et. al. TR-92-3, Department of Computer Science, WSU, 1992). Lemma 42 is the definition of Pythagoras difference. For Lemma 43, we introduce a new point M by construction (INTER M (PLINE A O U) (PLINE B O V)). Then by Lemma 22, $\overline{AB}^2 = \overline{AM}^2 + \overline{BM}^2$. By the second case of Lemma 39, $\frac{\overline{AM}}{\overline{OU}} = \frac{S_{AOBV}}{S_{OOUV}} = \frac{S_{AOV}-S_{BOV}}{S_{OUV}}$; $\frac{\overline{BM}}{\overline{OV}} = \frac{S_{AOU}-S_{BOU}}{S_{OUV}}$. We have Lemma 43. Lemma 44 is another basic fact taken for granted.

q.e.d.

Using Lemma 41 to 44, E and F can be written as expressions in $\overline{OU}$, $\overline{OV}$, and the area coordinates of the free points. Since the area coordinates of free points are independent, $E = F$ iff E and F are literally the same.

2.4.4 The Area Algorithm

→ $S = (C_1, C_2, \ldots, C_m, (E, F))$ is a statement in **C**.

← The algorithm tells whether S is true, or not, and if it is true, produces a proof for S.

```
for (i=m;i==1;i--) {
   if (the ndg conditions of Ci is satisfied) exit;
   // Let G1,\ldots,Gn be the geometric quantities in E and F
   for (j=1;j<=n,j++) {
      Hj <- eliminating the point introduced by construction Ci from Gj
      E <- E[Gj:=Hj]
      F <- F[Gj:=Hj]
   }
}
if (E==F) S <- true
else S<-false
```

The ndg condition of a construction has three forms: $A \neq B$, $PQ \nparallel UV$, or $PQ \not\perp UV$. For the first case we check if $P_{ABA} = 2\overline{AB}^2 = 0$. For the second case, we check if $S_{PUV} = S_{QUV}$. For the third case, we check if $P_{PUV} = P_{QUV}$. If a ndg condition of a geometry statement is not satisfied, the statement is trivially true.

Proof of the correctness. Only the last step needs explanation. If $E = F$, the statement is obviously true. Note that the ndg condition ensure that the denominators of all the expressions occurring in the proof do not vanish.

Otherwise, since the geometric quantities in E and F are all free parameters, i.e., in the geometric configuration of S they can take arbitrary values. Since $E \neq F$, we can take some concrete values for these quantities such that when replacing these quantities by the corresponding values in E and F, we obtain two different numbers. In other words, we obtain a counter example for S.

q.e.d.

In the next chapter we will describe an implementation of this method.

Chapter 3

Implementation of the area method within GCLC

GCLC (Djorić & Janičić, 2004; Janičić & Trajković, 2003) (from Geometry Constructions to LaTeX Converter) is a tool for producing mathematical illustrations and for teaching geometry. It basic functionality is converting formal descriptions of geometric constructions into digital figures.

The theorem prover built into **GCLC** is based on Chou's algorithm for proving geometry theorems (*area method*) as described in the last chapter. The proofs are expressed in terms of higher-level geometry lemmas and expression simplifications. The prover can prove a range of non-trivial theorems, including theorems due to Ceva, Menelaus, Gauss, Pappus, Tales etc (see appendix A).

Support for the prover involves only three commands: `prove` (for providing a conjecture), `prooflevel` (for setting the level of proof details), and `prooflimit` (for setting maximal size of a proof). The prover works in both command line version and in **WinGCLC** (and it does not use any specific functionalities of **WinGCLC**). Proofs of theorems are generated in LaTeX form and saved in a file. Each deduction step is accompanied by its semantics counterpart — corresponding numeric values in Cartesian plane.

The theorem prover is very efficient. Many conjectures are proved in only milliseconds. However, some conjecture may take several seconds, several minutes, or even several hours. The maximal number of proof steps can be set by the command `prooflimit`. The default value is 10000 proof steps.[1] If the prover perform more proof steps, the proving process is stopped.

3.1 Introductory Example

The theorem prover is tightly integrated into **GCLC** . This means that one can use the prover to reason about a **GCLC** construction (i.e., about objects introduced in it), without any required adaptations required for the deduction process. Of course, only the conjecture itself has to be added.

The example **GCLC** code given in Figure 3.1 describes a triangle and midpoints of two of triangle's sides. This **GCLC** code produces the figure 3.2. It holds that the lines AB and A_1B_1 are parallel and this can be proved by the theorem prover. The conjecture

[1] On a modern PC computer, 10000 steps are performed in less then 1 minute.

```
point A 20 10
point B 70 10
point C 35 40

midpoint B_1 B C
midpoint A_1 A C

drawsegment A B
drawsegment A C
drawsegment B C
drawsegment A_1 B_1

cmark_b A
cmark_b B
cmark_t C
cmark_l A_1
cmark_r B_1

prove { equal { signed_area3 A_1 B_1 A } { signed_area3 A_1 B_1 B } }
```

Figure 3.1: Description of a triangle and midpoints of two of triangle's sides and the conjecture of midpoint theorem

"AB and A_1B_1 are parallel" can (and has to be) stated in terms of "geometry quantities": $S_{A_1B_1A} = S_{A_1B_1B}$. This conjecture is given as argument to the `prove` command (in this case: `equal { signed_area3 A_1 B_1 A } { signed_area3 A_1 B_1 B }`). At the end of the processing of the **GCLC** file, the theorem prover is invoked; it produces a proof in LaTeX form in the file `name-proof.tex` (in the current directory, `name` is the name of the input file) and, within the **GCLC** report, a report about the proving process: whether the conjecture was proved, data about CPU time spent, and number of proof steps performed (in several categories).

3.2 Geometry Quantities and Stating Conjectures

The theorem prover deals with the geometry quantities described above, e.g. *ratios of directed segments*, *signed areas*, and *Pythagoras differences*. In **GCLC** , geometry quantities are written as in the following examples:

ratio of directed segments	$\frac{\overline{PQ}}{\overline{AB}}$	`sratio P Q A B`
signed area (arity 3)	S_{ABC}	`signed_area3 A B C`
signed area (arity 4)	S_{ABCD}	`signed_area4 A B C D`
Pythagoras difference (arity 3)	P_{ABC}	`pythagoras_difference3 A B C`
Pythagoras difference (arity 4)	P_{ABCD}	`pythagoras_difference4 A B C D`

A conjecture to be proved is given as argument to the `prove` command. It has to be of the form $L = R$. The conjecture can involve geometry quantities (only) over points already introduced (by a subset of commands) within the current construction.

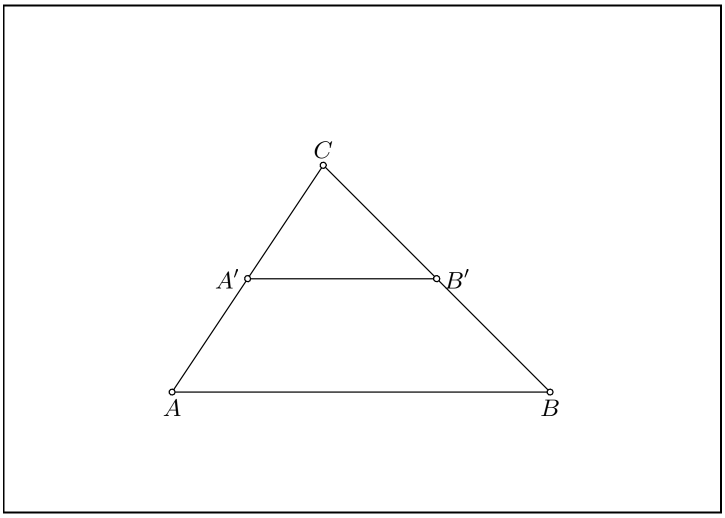


Figure 3.2: Illustration generated from the **GCLC** code from Figure 3.1

Geometry quantities can be combined together into more complex terms by operators for addition, multiplication and division. Operators are written in textual form as in the following table:

$=$	equality
$+$	sum
$\cdot$	mult
$/$	ratio

The conjecture and all its sub-terms are written in prefix form, with brackets if needed. For instance,

$$S_{A_1B_1A} = S_{A_1B_1B}$$

is given to be proved in the following way:

```
prove { equal { signed_area3 A_1 B_1 A }
              { signed_area3 A_1 B_1 B }
      }
```

and

$$\left(\left(\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}}\right) \cdot \frac{\overline{CE}}{\overline{EA}}\right) = 1$$

is given to be proved in the following way:

```
prove { equal { mult { mult { sratio A F F B }
                            { sratio B D D C } }
                     { sratio C E E A } }
              1 }
```

A range of geometry conjectures can be stated in terms of geometry quantities. Some of them are given in the following table.

points A and B are identical	iff	$P_{ABA} = 0$
points A, B, C are collinear	iff	$S_{ABC} = 0$
AB is perpendicular to CD	iff	$P_{ACD} = P_{BCD}$
AB is parallel to CD	iff	$S_{ACD} = S_{BCD}$
O is the midpoint of AB	iff	$\frac{\overline{AO}}{\overline{OB}} = 1$
AB has the same length as CD	iff	$P_{ABA} = P_{CDC}$
points A, B, C, D are harmonic	iff	$\frac{\overline{AC}}{\overline{CB}} = \frac{\overline{DA}}{\overline{DB}}$

The conjecture can involve geometry quantities only over points and lines already introduced within the current construction, and by using (only) the following commands:

- `point`
- `line`
- `intersec`
- `midpoint`
- `med`
- `perp`
- `foot`
- `parallel`
- `translate`
- `towards`
- `online`

The prover cannot prove conjectures about object constructed by using some other commands. For instance, if a line a is constructed by the command `bis`, then the prove cannot prove conjectures involving a or involving points constructed by using a.

3.3 Underlying Algorithm

The theorem prover is based on the algorithm described in the previous chapter. The basic idea of the algorithm is to express a theorem in terms of geometry quantities, to eliminate (by appropriate lemmas) all occurrences of constructed point and to simplify the expression, yielding a trivial equality.

3.3.1 Underlying Constructions

As we have stated in section 2.4.2 For each point X constructed, and for each geometry quantity g involving X, there is a suitable lemma that enables replacing g by an expression with no occurrences of X. Thanks to these lemmas, all constructed points can be eliminated from the conjecture.

3.3.2 Integration of Algorithm and Auxiliary Points

In order to be tightly integrated into **GCLC** , the prover uses standard **GCLC** construction commands and, if needed, transforms them internally into form required by the algorithm and/or introduces some auxiliary points:

`midpoint` is expressed in terms of `PRATIO`, it does not introduce new points;

`foot` is expressed in terms of `FOOT`, it does not introduce new points;

`med` introduces two auxiliary points: for instance, `med m A B` introduce a point M_m as the midpoint of AB and a point T_m on the bisector of AB (such that `TRATIO` $\mathtt{T_m}$ $\mathtt{M_m}$ `A 1`); the line m is then determined by the points M_m and T_m;

`perp` introduces one auxiliary point: if A lies on the line q, then `perp p A q` introduces a point T_p on a line perpendicular to q (such that `TRATIO` $\mathtt{T_p}$ `A` $\mathtt{Q_1}$ `1`; where the line q is determined by points Q_1 and Q_2); in this case, the line p is determined by the points A and T_p; if A does not lie on the line q, then `perp p A q` introduce a point F_p which is a foot of the normal from A to the line q; in this case, the line p is determined by the points A and F_p;

`parallel` introduces one auxiliary point: for instance, `parallel p A q` introduces a point P_p on a line parallel to q (such that `PRATIO` $\mathtt{P_p}$ `A` $\mathtt{Q_1}$ $\mathtt{Q_2}$ `1`; the line p is then determined by the points A and P_p;

`translate` is expressed in terms of `PRATIO`, it does not introduce new points;

`towards` is expressed in terms of `PRATIO`, it does not introduce new points;

`online` is expressed in terms of `PRATIO`, it does not introduce new points, but introduces a (indeterminate) constant r: for instance, `online X A B` is interpreted as `PRATIO X A A B r`.

Definitions of auxiliary points are given at the beginning of the proof.

3.3.3 Non-degenerative Conditions and Lemmas

Some constructions are possible only if certain conditions are met. For instance, the construction `inter X a b` is possible only if the lines `a` and `b` are not parallel. For such constructions *non-degenerative conditions* are store for future possible use and listed at the end of the proof.

Some non-degenerative conditions can also be introduced during the proving process:

- some lemmas have two cases (for instance, „if A belongs to CD“ and „if A does not belong to CD“); if a condition for one case can be proved (as a lemma), then that case is applied, otherwise, a condition for one case (the one of the form $L \neq R$) is assumed and introduced as a non-degenerative condition.
- in the cancellation rule, if all summands on both sides of the equality have the same multiplication factor X, the rule tries to prove (as a lemma) that $X = 0$; if this fails, a condition $X \neq 0$ is assumed and introduced as a non-degenerative condition and the equality is cancelled by X.

Lemmas are being proved as separate conjectures, but, of course, sharing the construction and non-degenerative conditions with outer context.

3.3.4 Structure of Algorithm

The algorithm has one main *while* loop — it process the sequence of all (relevant) constructions in backward manner (from last to first construction step) and transforms the current goal as follows:

- the current goal is initially the given conjecture;
- *while* there are construction steps do:
  - apply geometric simplifications to the current goal;
  - apply algebraic simplifications to the current goal;
  - if the current construction step introduce a new point P, then eliminate (using the elimination rules) one of occurrences of P (from the current goal) and go to the top of the while loop; otherwise, go to next construction step.
- apply geometric simplifications to the current goal;
- apply algebraic simplifications to the current goal.
- if the current goal is a trivial equation, then the conjecture has been proved, otherwise, the conjecture has not been proved.

The reasoning steps, as seen from the above overall algorithm, are divided into three groups:

algebraic simplifications: applies simplification rewrite rule (not directly related to geometry) such as:

$$\begin{array}{rcl} x+0 & \rightarrow & x \\ 0+x & \rightarrow & x \\ x\cdot 1 & \rightarrow & x \\ x\cdot 0 & \rightarrow & 0 \\ \dfrac{x}{y}+\dfrac{u}{v} & \rightarrow & \dfrac{x\cdot v+u\cdot y}{y\cdot v} \\ \ldots & & \end{array}$$

Algebraic simplifications are discussed in more details in §3.3.5.

geometric simplifications: applies simplification rewrite rule, directly related to geometry quantities such as:

$$\begin{array}{rcl} S_{AAB} & \rightarrow & 0 \\ S_{ABC} & \rightarrow & S_{BCA} \\ P_{AAB} & \rightarrow & 0 \\ \ldots & & \end{array}$$

Geometric simplifications are discussed in more details in §3.3.6.

elimination simplifications: applies elimination lemmas for eliminating constructed points for the current goal; for instance, if the point Y is introduced by as the intersection of lines l_1 (determined by U and V) and l_2 (determined by P and Q), then Y can be eliminated from expression of the form $\frac{\overline{AY}}{\overline{CD}}$ using the following equality:

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{S_{APQ}}{S_{CPDQ}}, & \text{if } A \in UV \\ \frac{S_{AUV}}{S_{CUDV}}, & \text{if } A \notin UV \end{cases}$$

The lemmas 8 and 29 through 40 are used for elimination simplifications. Note that some lemmas have two cases, one of them is chosen in the manner described in §3.3.3.

3.3.5 Algebraic Simplification

The author of the paper (Chou *et al.*, 1993) do not discuss the important issue of simplification of expressions, but only geometrical aspects and elimination lemmas. However, this is not sufficient for implementing the prover based on the area method. In this part of the paper, we give a full description of the algebraic simplification used in our implementation of the area method. Algebraic simplifications are based on rewrite rules, with some of them conditional rewrite rules and with some of them that cannot be described as first-order rules. The following rules are used:

Multiplication by zero:

$$\begin{aligned} x \cdot 0 &\rightarrow 0 \\ 0 \cdot x &\rightarrow 0 \end{aligned}$$

Multiplication by one:

$$\begin{aligned} x \cdot 1 &\rightarrow x \\ 1 \cdot x &\rightarrow x \end{aligned}$$

Zero fraction up:

$$\frac{0}{x} \rightarrow 0$$

Fraction equal zero:

$$\frac{x}{y} = 0 \rightarrow x = 0$$

Summation with zero:

$$\begin{aligned} x + 0 &\rightarrow x \\ 0 + x &\rightarrow x \end{aligned}$$

Multiplication of constants:

$c_1 \cdot c_2 \rightarrow c_3$ where c_1 and c_2 are constants (constant real numbers) and $c_1 \cdot c_2 = c_3$

Ratio cancellation:

$$\begin{array}{rcl} \dfrac{\overline{AB}}{\overline{AB}} & \rightarrow & 1 \\ \dfrac{\overline{AB}}{\overline{BA}} & \rightarrow & -1 \\ \dfrac{E_1 \cdot \ldots \cdot E_{i-1} \cdot \ldots \cdot C \cdot E_{i+1} \cdot \ldots \cdot E_n}{E'_1 \cdot \ldots \cdot E'_{j-1} \cdot \ldots \cdot C \cdot E'_{j+1} \cdot \ldots \cdot E'_m} & \rightarrow & \dfrac{E_1 \cdot \ldots \cdot E_{i-1} \cdot E_{i+1} \cdot \ldots \cdot E_n}{E'_1 \cdot \ldots \cdot E'_{j-1} \cdot E'_{j+1} \cdot \ldots \cdot E'_m} \end{array}$$

Similar summands:

$$\begin{array}{rcl} E_1 + \cdots + E_{i-1} + c_1 \cdot C + E_{i+1} + & \cdots & + E_{j-1} + c_2 \cdot C' + E_{j+1} + \cdots + E_n \\ & \rightarrow & \\ E_1 + \cdots E_{i-1} + c_3 \cdot C + E_{i+1} + & \cdots & + E_{j-1} + 0 + E_{j+1} + \cdots + E_n \end{array}$$

where c_1 and c_2 are constants (constant real numbers) and $c_1 \cdot c_2 = c_3$ and C and C' are equal products (with all multiplicands equal up to permutation). Note that this rule assumes that multiplication and addition are commutative and associative.

Similar summands on two sides:

$$\begin{array}{rcl} E_1 + \cdots + E_{i-1} + c_1 \cdot C + E_{i+1} + \cdots + E_n & = & E'_1 + \cdots + E'_{j-1} + c_2 \cdot C' + E'_{j+1} + \cdots + E'_m \\ & \rightarrow & \\ E_1 + \cdots + E_{i-1} + c_3 \cdot C + E_{i+1} + \cdots + E_n & = & E'_1 + \cdots + E'_{j-1} + 0 + E'_{j+1} + \cdots + E'_m \end{array}$$

where c_1 and c_2 are constants (constant real numbers) and $c_1 - c_2 = c_3$ and C and C' are equal products (with all multiplicands equal up to permutation). Note that this rule assumes that multiplication and addition are commutative and associative.

Fraction with constant numerator:

$$\begin{array}{rcl} \dfrac{x}{1} & \rightarrow & x \\ \dfrac{x}{c} & \rightarrow & (1/c) \cdot x \end{array}$$

where c is a constant (constant real number) and $c \neq 1$

Multiple fraction:

$$\begin{array}{rcl} \dfrac{a}{\frac{b}{c}} & \rightarrow & \dfrac{a \cdot c}{b} \\ \dfrac{\frac{a}{b}}{c} & \rightarrow & \dfrac{a}{b \cdot c} \\ \dfrac{\frac{a}{b}}{\frac{c}{d}} & \rightarrow & \dfrac{a \cdot d}{c \cdot b} \end{array}$$

Multiplication of fractions:

$$
\begin{array}{rcl}
a \cdot \dfrac{b}{c} & \rightarrow & \dfrac{a \cdot b}{c} \\
\dfrac{a}{b} \cdot c & \rightarrow & \dfrac{a \cdot c}{b} \\
\dfrac{a}{b} \cdot \dfrac{c}{d} & \rightarrow & \dfrac{a \cdot c}{b \cdot d}
\end{array}
$$

Commutativity with number:

$$x \cdot c \quad \rightarrow \quad c \cdot x$$

where c is a constant (constant real number) and x is not a constant.

Associativity and commutativity:

$$x \cdot (c \cdot y) \quad \rightarrow \quad c \cdot (x \cdot y)$$

where c is a constant (constant real number) and x is not a constant.

Cancellation: If the current goal is of the form

$$E_1 + \cdots E_{i-1} \cdots C + E_{i+1} + \cdots + E_n = E'_1 + \cdots E'_{j-1} \cdots C + E'_{j+1} + \cdots + E'_m$$

and if all summands E_i and E'_j have a common multiplication factor X, then try to prove that it holds $X = 0$:

- if $X = 0$ has been proved, the current goal can be rewritten to $0 = 0$;
- if $X = 0$ has been disproved (i.e., if $X \neq 0$ has been proved), then both sides in the current goal can be cancelled by X;
- if neither $X = 0$ nor $X \neq 0$ can be proved, then assume $X \neq 0$ (and add to the list of non-degenerative conditions) and cancel both sides in the current goal by X.

Note that this steps includes proving subgoals (which initiate the whole proving process on the new goal). However, note that there is no branching, so the proof is always sequential. See also §3.3.3.

Right associativity:

$$((a \cdot b) \cdot c) \quad \rightarrow \quad a \cdot (b \cdot c)$$

Distributivity over addition:

$$
\begin{array}{rcl}
a \cdot (b + c) & \rightarrow & a \cdot b + a \cdot c \\
(b + c) \cdot a & \rightarrow & b \cdot a + c \cdot a
\end{array}
$$

Sum of fractions:

$$
\begin{array}{rcl}
\dfrac{a}{b} + \dfrac{c}{b} & \rightarrow & \dfrac{a + c}{b} \\
\dfrac{a}{b} + c & \rightarrow & \dfrac{a + c \cdot b}{b} \\
c + \dfrac{a}{b} & \rightarrow & \dfrac{c \cdot b + a}{b} \\
\dfrac{a}{b} + \dfrac{c}{d} & \rightarrow & \dfrac{a \cdot d + c \cdot b}{bd}
\end{array}
$$

Fractions on two sides:

$$\begin{array}{rcl} \frac{a}{b} = \frac{c}{b} & \rightarrow & a = c \\ \frac{a}{b} = \frac{c}{d} & \rightarrow & a \cdot d = c \cdot b \end{array}$$

Elimination of fraction:

$$\begin{array}{rcl} \frac{a}{b} = c & \rightarrow & a = b \cdot c \\ a = \frac{c}{d} & \rightarrow & a \cdot d = c \end{array}$$

One side:

$$a = c \quad \rightarrow \quad a - c = 0$$

In the algebraic simplification step, these rules are used in the "waterfall" manner: they are tried for applicability, and when one rule is (once) applied successfully, then the list of the rules is tried from the top. The algebraic rules are ordered as above. It is not difficult to prove that such transformation process terminates.

3.3.6 Geometric Simplifications

Geometric simplifications are based on the following lemmas: 1, 2, 4, 10, 11, 17, 18, 19 and on the following definitions: 4, 5, 6. They are applied within the following rules:

Zero-valued geometry quantity:

$$\begin{array}{rcll} \overline{AA} & \rightarrow & 0 & \\ \frac{\overline{AA}}{\overline{BC}} & \rightarrow & 0 & \text{Lemma 11} \\ S_{ABC} & \rightarrow & 0 & \text{if } A, B, C \text{ are collinear; Lemma 2} \\ P_{AAB} & \rightarrow & 0 & \text{Lemma 17} \\ P_{BAA} & \rightarrow & 0 & \text{Lemma 17} \end{array}$$

Matching quantity:

- If there is a term $\overline{AB}$ in the current goal, apply exhaustively the following rewrite rule:

$$\overline{BA} \quad \rightarrow \quad \overline{AB}$$

- If there is a term S_{ABC} in the current goal, apply exhaustively the following rewrite rules:

$$\begin{array}{rcll} S_{BCA} & \rightarrow & S_{ABC} & \text{Lemma 1} \\ S_{CAB} & \rightarrow & S_{ABC} & \text{Lemma 1} \\ S_{ACB} & \rightarrow & -1 \cdot S_{ABC} & \text{Lemma 1} \\ S_{BAC} & \rightarrow & -1 \cdot S_{ABC} & \text{Lemma 1} \\ S_{CBA} & \rightarrow & -1 \cdot S_{ABC} & \text{Lemma 1} \end{array}$$

- If there is a term P_{ABC} in the current goal, apply exhaustively the following rewrite rule:

$$P_{CBA} \quad \rightarrow \quad P_{ABC} \qquad \text{Lemma 18}$$

- If there is a term P_{ABA} in the current goal, apply exhaustively the following rewrite rule:

$$P_{BAB} \quad \rightarrow \quad P_{ABA} \qquad \text{Lemma 19}$$

This transformation is performed in hope that some expression can be cancelled afterwards.

Orient: This rule is applied when the current construction step[2] introduces the point Y. Elimination lemmas enable eliminating a point from expressions only at certain positions — usually the last position in the list of the arguments. That is why it is necessary to transform relevant terms in the current goal. For terms S_{ABCD} and P_{ABCD} (of arity 4), the first step is to transform them into terms of arity 3.

$$\begin{array}{lcrl}
\frac{\overline{YA}}{\overline{BC}} & \rightarrow & -1 \cdot \frac{\overline{AY}}{\overline{BC}} & \text{Lemma 10} \\
\frac{\overline{BC}}{\overline{YA}} & \rightarrow & -1 \cdot \frac{\overline{BC}}{\overline{AY}} & \text{Lemma 10} \\
S_{AYB} & \rightarrow & S_{BAY} & \text{Lemma 1} \\
S_{YAB} & \rightarrow & S_{ABY} & \text{Lemma 1} \\
P_{YAB} & \rightarrow & P_{BAY} & \text{Lemma 18} \\
S_{YABC} & \rightarrow & S_{YAB} + S_{YBC} & \text{Definition 4} \\
S_{AYBC} & \rightarrow & S_{AYB} + S_{ABC} & \text{Definition 4} \\
S_{ABYC} & \rightarrow & S_{ABY} + S_{AYC} & \text{Definition 4} \\
P_{ABCD} & \rightarrow & P_{ABD} + -1 \cdot P_{CBD} & \text{Definition 6}
\end{array}$$

Geometric simplification is applied in each iteration of the *while*-loop in the algorithm, and then after the *while*-loop. In this last application, apart from the above rules, there are several additional ones, described below.

S4 to S3:

$$S_{ABCD} \quad \rightarrow \quad S_{ABC} + S_{ACD} \quad \text{Definition 4}$$

P4 to P3:

$$P_{ABCD} \quad \rightarrow \quad P_{ABD} + -1 \cdot P_{CBD} \quad \text{Definition 6}$$

H4 points:

$$S_{ABC} \quad \rightarrow \quad S_{ABD} + S_{ADC} + S_{DBC} \quad \text{Lemma 4}$$

if there are terms S_{ABD}, S_{ADC}, S_{DBC} in the current goal.

P3 to segments:

$$P_{ABC} \quad \rightarrow \quad AB^2 + CB^2 + -1 \cdot AC^2 \quad \text{Definition 5}$$

In the main loop, only the first three of the above rules are used.

In the geometric simplification step, the above rules are used in the "waterfall" manner: they are tried for applicability, and when one rule is (once) applied successfully, then the list of the rules is tried from the top. The geometric rules are ordered as above. It is not difficult to prove that such transformation process terminates.

[2] Recall that construction steps are processed one by one, in reversed order — from the last to the first one.

3.3.7 Scope

The theorem prover can prove *any* geometry theorem expressed in terms of geometry quantities, and involving only points introduced by using the commands `point`, `line`, `intersec`, `midpoint`, `med`, `perp`, `foot`, `parallel`, `translate`, `towards`, `online`. This can be proved following the ideas from (Chou *et al.*, 1993). However, some of the proofs can be very long.

3.4 Prover Output

A proof is exported in LaTeX form using a special style file `gclc.sty`, in the file `name-proof.tex` (in the current directory, `name` is the name of the input file). If there is no `prove` command within the construction, then the file `name-proof.tex` will not be created.

At the beginning of the proof, the auxiliary points are defined, for instance:

Let M_a^0 be the midpoint of the segment BC. Let T_a^1 be the point on bisector of the segment BC (such that `TRATIO` T_a^1 M_a^0 B 1).

The proof consists of *proof steps*. In each proof step, the current goal is changed. For each proof step, there is an explanation and its semantics counterpart. This semantic information is calculated for concrete points used in the construction (note that these coordinates are never used in the proof itself); it can serve as a semantic test, especially for conjectures for which is not known whether or not they are theorems. Proof step are enumerated. For example:

$\left(\left(\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}}\right) \cdot \frac{\overline{CE}}{\overline{EA}}\right)$	$= 1$	by the statement	(1)
$\left(\left(\left(-1 \cdot \frac{\overline{AF}}{\overline{BF}}\right) \cdot \frac{\overline{BD}}{\overline{DC}}\right) \cdot \frac{\overline{CE}}{\overline{EA}}\right)$	$= 1$	by geometric simplifications	(2)

The gclc style has three options controlling the output, the above example show the default values, e.g. `portrait`, `small` and no semantic values. The options and its values are:

orientation & style `portrait` – uses the package *longtable* to generate a multi-page table (default value); `portraitbreqn` – uses the package *breqn* to try to break automatically the equations. Notice that the package *breqn* fails if it encounter extra large fractions (it fails if the fraction is larger then the `textwidth`); `landscape` - uses the packages *lscape*, *amsmath* (with option *leqno*), and *breqn*, to generate the list of equations in landscape mode, with numbers on the left, and with automatic equation breaking.

size The normal size names of LaTeX, from "tiny" up to "large" are used in order to define the size of the fonts inside a demonstration. The default value is `small`.

semantics if used, it displays the semantic values of both sides of equations. The default value is `NULL`.

Lemmas are proved within the main proof (making nested proof levels), and the beginning and the end of a proof for a lemma is marked by a horizontal solid line.

At the end of a proof, it is reported if the conjecture is proved ("Q.E.D." — lat. Quod Erat Demonstrandum — which was required to prove) or not.

At the end of the main proof all non-degenerative conditions are listed. For instance:

$S_{M_a^0 M_b^2 T_b^3} \neq S_{T_a^1 M_b^2 T_b^3}$ i.e., lines $M_a^0 T_a^1$ and $M_b^2 T_b^3$ are not parallel (construction based assumption)

See some complete examples in the appendix.

3.4.1 Controlling Level of Output

The level of proof output is controlled by the command `prooflevel`. This command has one argument (an integer from 0 to 7) which provides the output level:

0 : no output (except the statement);

1 : elimination steps plus grouped geometric steps and algebraic steps;

2 : elimination steps plus geometric steps plus grouped algebraic steps;

3 : as level 2, plus statements of lemmas;

4 : as level 3, plus elimination steps plus grouped geometric steps and algebraic steps in lemmas;

5 : as level 4, plus geometric steps in lemmas;

6 : as level 5, plus algebraic steps at proof level 0;

7 : as level 6, plus algebraic steps in lemmas.

The default output level is 1.

3.4.2 Prover Short Report

Apart from the proof exported to file `name-proof.tex`, the prover produces a short report (if there was a conjecture given in the GCL file). In the command line version, this short report is shown and written in the log file, while **WinGCLC** shows this report in its output window. The report consists of information on number of steps performed, on CPU time spent and whether or not the conjecture has been proved. For example:

```
Number of elimination proof steps:      3
Number of geometric proof steps:        6
Number of algebraic proof steps:       23
Total number of proof steps:           32

Time spent by the prover: 0.002 seconds
The conjecture successfully proved.
The prover output is written in the file ceva-proof.tex.
```

Appendix A

Proofs examples

A.1 GeoThms - Geometry Theorems Database

As a support for the other programs there exist the *GeoThms* data base. The *GeoThms - Geometry Theorems Database* aims to provide a common repository of theorems and proofs in the area of constructive problems in Euclidean Geometry, in particular for the *Area Method*.

The examples described next, along with many others can be found in:

`http://hilbert.mat.uc.pt/~geothms/`

using as user and password "`anonymous`".

A.2 Proof of Ceva's Theorem

A.2.1 Default style

The complete proof compiled with the default options.

$$\left(\left(\frac{\overrightarrow{AF}}{\overrightarrow{FB}}\cdot\frac{\overrightarrow{BD}}{\overrightarrow{DC}}\right)\cdot\frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right) = 1 \qquad \text{by the statement} \tag{0}$$

$$\left(\left(\left(-1\cdot\frac{\overrightarrow{AF}}{\overrightarrow{BF}}\right)\cdot\frac{\overrightarrow{BD}}{\overrightarrow{DC}}\right)\cdot\frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right) = 1 \qquad \text{by geometric simplifications} \tag{1}$$

$$\left(-1\cdot\left(\frac{\overrightarrow{AF}}{\overrightarrow{BF}}\cdot\left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}}\cdot\frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right)\right)\right) = 1 \qquad \text{by algebraic simplifications} \tag{2}$$

$$\left(-1\cdot\left(\frac{S_{APC}}{S_{BPC}}\cdot\left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}}\cdot\frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right)\right)\right) = 1 \qquad \text{by Lemma 8 (point } F \text{ eliminated)} \tag{3}$$

$$\left(-1\cdot\left(\frac{S_{APC}}{S_{BPC}}\cdot\left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}}\cdot\left(-1\cdot\frac{\overrightarrow{CE}}{\overrightarrow{AE}}\right)\right)\right)\right) = 1 \qquad \text{by geometric simplifications} \tag{4}$$

$$\frac{\left(S_{APC}\cdot\left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}}\cdot\frac{\overrightarrow{CE}}{\overrightarrow{AE}}\right)\right)}{S_{BPC}} = 1 \qquad \text{by algebraic simplifications} \tag{5}$$

$\frac{\left(S_{APC}\cdot\left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}}\cdot\frac{S_{CPB}}{S_{APB}}\right)\right)}{S_{BPC}}$	=	1	by Lemma 8 (point E eliminated)	(6)
$\frac{\left(S_{APC}\cdot\left(\left(-1\cdot\frac{\overrightarrow{BD}}{\overrightarrow{CD}}\right)\cdot\frac{S_{CPB}}{S_{APB}}\right)\right)}{(-1\cdot S_{CPB})}$	=	1	by geometric simplifications	(7)
$\frac{\left(S_{APC}\cdot\frac{\overrightarrow{BD}}{\overrightarrow{CD}}\right)}{S_{APB}}$	=	1	by algebraic simplifications	(8)
$\frac{\left(S_{APC}\cdot\frac{S_{BPA}}{S_{CPA}}\right)}{S_{APB}}$	=	1	by Lemma 8 (point D eliminated)	(9)
$\frac{\left(S_{APC}\cdot\frac{S_{BPA}}{(-1\cdot S_{APC})}\right)}{(-1\cdot S_{BPA})}$	=	1	by geometric simplifications	(10)
1	=	1	by algebraic simplifications	(11)

Q.E.D.

NDG conditions are:

$S_{BPA} \neq S_{CPA}$ i.e., lines BC and PA are not parallel (construction based assumption)

$S_{APB} \neq S_{CPB}$ i.e., lines AC and PB are not parallel (construction based assumption)

$S_{APC} \neq S_{BPC}$ i.e., lines AB and PC are not parallel (construction based assumption)

$P_{FBF} \neq 0$ i.e., points F and B are not identical (conjecture based assumption)

$P_{DCD} \neq 0$ i.e., points D and C are not identical (conjecture based assumption)

$P_{EAE} \neq 0$ i.e., points E and A are not identical (conjecture based assumption)

Number of elimination proof steps: 3

Number of geometric proof steps: 6

Number of algebraic proof steps: 23

Total number of proof steps: 32

Time spent by the prover: 0.001 seconds

A.2.2 Landscape & Semantics

A fragment of the proof compiled with the options `landscape` and `semantics`.

$$\left(\left(\frac{\overrightarrow{AF}}{\overrightarrow{FB}} \cdot \frac{\overrightarrow{BD}}{\overrightarrow{DC}}\right) \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right) = 1 \quad \text{, by the statement (value 1=1)} \tag{1}$$

$$\left(\left(\left(-1 \cdot \frac{\overrightarrow{AF}}{\overrightarrow{BF}}\right) \cdot \frac{\overrightarrow{BD}}{\overrightarrow{DC}}\right) \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right) = 1 \quad \text{, by geometric simplifications (value 1=1)} \tag{2}$$

$$\left(-1 \cdot \left(\frac{\overrightarrow{AF}}{\overrightarrow{BF}} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right)\right)\right) = 1 \quad \text{, by algebraic simplifications (value 1=1)} \tag{3}$$

$$\left(-1 \cdot \left(\frac{S_{APC}}{S_{BPC}} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right)\right)\right) = 1 \quad \text{, by Lemma 8 (point } F \text{ eliminated) (value 1=1)} \tag{4}$$

$$\left(-1 \cdot \left(\frac{S_{APC}}{S_{BPC}} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \left(-1 \cdot \frac{\overrightarrow{CE}}{\overrightarrow{AE}}\right)\right)\right)\right) = 1 \quad \text{, by geometric simplifications (value 1=1)} \tag{5}$$

$$\frac{\left(S_{APC} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{AE}}\right)\right)}{S_{BPC}} = 1 \quad \text{, by algebraic simplifications (value 1=1)} \tag{6}$$

$$\frac{\left(S_{APC} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{S_{CPB}}{S_{APB}}\right)\right)}{S_{BPC}} = 1 \quad \text{, by Lemma 8 (point } E \text{ eliminated) (value 1=1)} \tag{7}$$

$$\frac{\left(S_{APC} \cdot \left(\left(-1 \cdot \frac{\overrightarrow{BD}}{\overrightarrow{CD}}\right) \cdot \frac{S_{CPB}}{S_{APB}}\right)\right)}{(-1 \cdot S_{CPB})} = 1 \quad \text{, by geometric simplifications (value 1=1)} \tag{8}$$

A.3 Harmonic Set Proof

A.3.1 Portrait with Automatic Breaking of Equations

A fragment of the proof compiled with the option `portraitbreqn`.

$$\frac{\overrightarrow{LF}}{\overrightarrow{KF}} = \frac{\overrightarrow{LG}}{\overrightarrow{GK}}, \quad \text{by the statement} \tag{1}$$

$$\frac{\overrightarrow{LF}}{\overrightarrow{KF}} = \left(-1 \cdot \frac{\overrightarrow{LG}}{\overrightarrow{KG}}\right), \quad \text{by geometric simplifications} \tag{2}$$

$$\frac{\overrightarrow{LF}}{\overrightarrow{KF}} = \left(-1 \cdot \frac{S_{LAC}}{S_{KAC}}\right), \quad \text{by Lemma 8 (point } G \text{ eliminated)} \tag{3}$$

$$\frac{\overrightarrow{LF}}{\overrightarrow{KF}} = \frac{(-1 \cdot S_{LAC})}{S_{KAC}}, \quad \text{by algebraic simplifications} \tag{4}$$

$$\frac{S_{LBD}}{S_{KBD}} = \frac{(-1 \cdot S_{LAC})}{S_{KAC}}, \quad \text{by Lemma 8 (point } F \text{ eliminated)} \tag{5}$$

$$(S_{LBD} \cdot S_{KAC}) = (-1 \cdot (S_{LAC} \cdot S_{KBD})), \quad \text{by algebraic simplifications} \tag{6}$$

$$(S_{LBD} \cdot S_{ACK}) = (-1 \cdot (S_{LAC} \cdot S_{BDK})), \quad \text{by geometric simplifications} \tag{7}$$

$$\left(S_{LBD} \cdot \frac{((S_{ABC} \cdot S_{ACD}) + (-1 \cdot (S_{DBC} \cdot S_{ACA})))}{S_{ABDC}}\right) = (-1 \cdot (S_{LAC} \cdot S_{BDK})), \quad \text{by Lemma 30 (point } K \text{ eliminated)} \tag{8}$$

$$\left(S_{LBD} \cdot \frac{((S_{ABC} \cdot S_{ACD}) + (-1 \cdot (S_{DBC} \cdot 0)))}{S_{ABDC}}\right) = (-1 \cdot (S_{LAC} \cdot S_{BDK})), \quad \text{by geometric simplifications} \tag{9}$$

$$\frac{(S_{LBD} \cdot (S_{ABC} \cdot S_{ACD}))}{S_{ABDC}} = (-1 \cdot (S_{LAC} \cdot S_{BDK})), \quad \text{by algebraic simplifications} \tag{10}$$

$$\frac{(S_{LBD} \cdot (S_{ABC} \cdot S_{ACD}))}{S_{ABDC}} = \left(-1 \cdot \left(S_{LAC} \cdot \frac{((S_{ABC} \cdot S_{BDD}) + (-1 \cdot (S_{DBC} \cdot S_{BDA})))}{S_{ABDC}}\right)\right), \quad \text{by Lemma 30 (point } K \text{ eliminated)} \tag{11}$$

$$\frac{(S_{LBD} \cdot (S_{ABC} \cdot S_{ACD}))}{S_{ABDC}} = \left(-1 \cdot \left(S_{LAC} \cdot \frac{((S_{ABC} \cdot 0) + (-1 \cdot (S_{DBC} \cdot S_{BDA})))}{S_{ABDC}}\right)\right), \quad \text{by geometric simplifications} \tag{12}$$

$$(S_{LBD} \cdot (S_{ABC} \cdot S_{ACD})) = (S_{LAC} \cdot (S_{DBC} \cdot S_{BDA})), \quad \text{by algebraic simplifications} \tag{13}$$

$$(S_{BDL} \cdot (S_{ABC} \cdot S_{ACD})) = (S_{ACL} \cdot (S_{DBC} \cdot S_{BDA})), \quad \text{by geometric simplifications} \tag{14}$$

$$\left(\frac{((S_{ACD} \cdot S_{BDB}) + (-1 \cdot (S_{BCD} \cdot S_{BDA})))}{S_{ACBD}} \cdot (S_{ABC} \cdot S_{ACD})\right) = (S_{ACL} \cdot (S_{DBC} \cdot S_{BDA})), \quad \text{by Lemma 30 (point } L \text{ eliminated)} \tag{15}$$

$$\left(\frac{((S_{ACD} \cdot 0) + (-1 \cdot (S_{BCD} \cdot S_{BDA})))}{S_{ACBD}} \cdot (S_{ABC} \cdot S_{ACD})\right) = (S_{ACL} \cdot (S_{BCD} \cdot S_{BDA})), \quad \text{by geometric simplifications} \tag{16}$$

$$\frac{(-1 \cdot (S_{BCD} \cdot (S_{BDA} \cdot (S_{ABC} \cdot S_{ACD}))))}{S_{ACBD}} = (S_{ACL} \cdot (S_{BCD} \cdot S_{BDA})), \quad \text{by algebraic simplifications} \tag{17}$$

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