Levi-parallel contact Riemannian manifolds

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joint work with Antonio Lotta

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Let \((M, \eta)\) be a contact manifold of dimension \(N = 2n + 1, n \geq 1\), i.e. \(\eta\) is a 1-form satisfying

\[\eta \wedge (d\eta)^n \neq 0\]

everywhere on \(M\).

Let \(D := \text{Ker}(\eta)\) be the contact distribution.
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The skew-symmetric Levi-Tanaka form

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L_{\eta}(X, Y) = -d\eta(X, Y) \quad X, Y \in D
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is nondegenerate.
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The skew-symmetric Levi-Tanaka form
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L_\eta(X, Y) = -d\eta(X, Y) \quad X, Y \in D
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is nondegenerate.

The Reeb vector field is the unique globally defined vector field $\xi$, transverse to $D$, such that
\[
\eta(\xi) = 1, \quad d\eta(X, \xi) = 0 \quad \text{for any } X \in \mathfrak{X}(M).
\]
An associated metric is a Riemannian metric $g$ for which there exists a $(1, 1)$-tensor field $\varphi$ such that

$$\varphi^2 = -Id + \eta \otimes \xi, \quad \eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \varphi Y).$$
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Equivalently,

- $\xi = \eta^\sharp$,
- the endomorphism $J := (L_\eta)^\sharp : D \to D$ defined by
  $$g(X, Y) = L_\eta(X, JY), \quad X, Y \in D$$
satisfies
  $$J^2 = -Id$$
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$(\varphi, \xi, \eta, g)$ is a contact metric structure

$J$ is integrable $\leadsto (M, J, \eta)$ is a pseudohermitian manifold

$J$ is integrable $+ \xi$ is Killing $\leadsto (\varphi, \xi, \eta, g)$ is a Sasakian structure
Curvature of associated metrics

Theorem (Blair, 1976)

A contact manifold of dimension $\geq 5$ cannot admit any flat associated Riemannian metric.
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Let $g$ be an associated metric on $(M, \eta)$ with constant sectional curvature $c$. If $\dim M \geq 5$ then $c = 1$ and the structure is Sasakian.
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Theorem (Boeckx-Cho, 2006)

A locally symmetric contact metric manifold of dimension \( N = 2n + 1 \) is either Sasakian of constant curvature 1 or locally isometric to \( E^{n+1} \times S^n(4) \).
Blair’s conjecture:

non existence of contact metric manifolds having **nonpositive curvature**, with the exception of the flat 3-dimensional case
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A compact contact manifold cannot admit any associated metric of negative curvature.

Proposition
Let \((M, \varphi, \xi, \eta, g)\) be a contact metric manifold such that \(\xi\) belongs to a \((\kappa, \mu)\)-nullity distribution, i.e.

\[
R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \quad h := \frac{1}{2}\mathcal{L}_\xi\varphi.
\]

Then \(M\) cannot have negative curvature.
Blair’s conjecture in the homogeneous case

A contact metric manifold is defined to be \textit{homogeneous} if it admits a transitive Lie group of diffeomorphisms preserving the structure tensor fields \((\varphi, \xi, \eta, g)\).
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**Theorem (Lotta, 2010)**

Let \((M, \varphi, \xi, \eta, g)\) be a simply connected *homogeneous* contact metric manifold having nonpositive sectional curvature. Then \(M\) is 3-dimensional, flat, and it is equivalent to the Lie group \(\tilde{E}(2)\), endowed with a left invariant contact metric structure.
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Actually, a more general result holds, involving *admissible* metrics instead of *associated* metrics.
Definition

Let \((M, \eta)\) be a contact manifold. A Riemannian metric \(g\) on \(M\) will be called admissible if

\[ \xi = \eta^\# \]

i.e. the Reeb vector field \(\xi\) is of unit length and orthogonal to the contact distribution \(D\) with respect to \(g\).

Every associated metric is admissible.
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Every associated metric is admissible.

Theorem (Lotta, 2010)

Let $(M, \eta)$ be a simply connected homogeneous contact manifold of dimension $N \geq 5$. Then $M$ does not admit any admissible homogeneous Riemannian metric $g$ having nonpositive curvature.
Question: do the curvature rigidity results also hold for admissible metrics? (not necessarily associated)
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(not necessarily associated)

We shall give an answer within the class of **Levi-parallel** contact Riemannian manifolds.
Proposition

Let \((M, \eta, g)\) be a contact manifold endowed with an admissible metric. Then there exists a unique connection \(\tilde{\nabla}\) on \(M\) such that

1. \(\tilde{\nabla}\eta = 0;\)
2. \(\tilde{\nabla}g = 0;\)
3. \(g(\tilde{T}(X,Y), Z) = 0\) for any \(X, Y, Z \in \mathcal{D};\)
4. the tensor \(\tau: \mathcal{D} \to \mathcal{D}\) defined by

\[
\tau X = \tilde{T}(\xi, X) \quad \forall X \in \mathcal{D}
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is symmetric with respect to \(g\).

\(\tilde{\nabla}\) will be called the canonical connection associated to \((\eta, g)\).

It induces a connection on the vector bundle \(\mathcal{D}\).
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is symmetric with respect to $g$.

$\tilde{\nabla}$ will be called the canonical connection associated to $(\eta, g)$. It induces a connection on the vector bundle $\mathcal{D}$.

Significance of $\tau$:

$$\tau = 0 \iff \xi \text{ is Killing.}$$
If \( g \) is an associated metric then \( \tilde{\nabla} \) is the Tanno connection.

For pseudohermitian manifolds it is the Tanaka-Webster connection.
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**Definition**

We say that $(M, \eta, g)$ is a **Levi-parallel contact Riemannian manifold** if the Levi-Tanaka form $L_\eta$ is parallel with respect to $\tilde{\nabla}$.

*If moreover $\xi$ is Killing, we say that $(M, \eta, g)$ is of **Sasakian type**.*
Let \((M, \eta, g)\) be a contact manifold with a fixed admissible metric. Let

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be the endomorphism dual of \(L_\eta\) with respect to \(g\).
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Extend \(\varphi\) to a skew-symmetric \((1, 1)\)-tensor field requiring \(\varphi(\xi) = 0\), so that

\[ d\eta(X, Y) = g(X, \varphi Y) \quad X, Y \in \mathfrak{X}(M). \]
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The following are equivalent:
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- \((\nabla_X \varphi)Y = g(\varphi(\tau - \varphi)X, Y)\xi - \eta(Y)\varphi(\tau - \varphi)X\).
- \(\nabla \xi \varphi = 0\) and \(\varphi\) is \(\eta\)-parallel

\(\varphi\) is \(\eta\)-parallel \iff \(g((\nabla_X \varphi)Y, Z) = 0\) for every \(X, Y, Z \in D\).
If $g$ is an associated metric on $(M, \eta)$, by a result of Tanno (1989), the following are equivalent:

- $(M, \eta, g)$ is Levi-parallel,
- the almost $CR$ structure $(D, J := \varphi|_D)$ is integrable.
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Consequence of the parallelism:

Let $(M, \eta, g)$ be a Levi-parallel contact Riemannian manifold. The spectrum $\mathcal{S}$ of the symmetric operator

$$\varphi^2 : D \rightarrow D$$

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consists of negative constants.

For $\lambda \in S$, denote by $D(\lambda) \subset D$ the eigendistribution of $\varphi^2$ with eigenvalue $\lambda$.

It has even rank, it is $\tilde{\nabla}$-parallel and invariant under $\tau$ and $\varphi$. 
Basic example with prescribed Levi-Tanaka form:

Let \((V, \langle \cdot, \cdot \rangle)\) be a Euclidean vector space, \(\Theta : V \times V \to \mathbb{R}\) a symplectic form.

On the space \(m := V \oplus \mathbb{R}\) define a nilpotent Lie algebra structure:

\[ [X,Y] := 2\Theta(X,Y) \quad X,Y \in V. \]

Extend \(\langle \cdot, \cdot \rangle\) in a natural way to a scalar product \(g\) on \(m\).

Let \(\eta\) be the 1-form on \(m\) such that \(\eta(X + t) = t\).

Let \(M\) is simply connected Lie group with \(\text{Lie}(M) = m\) carries a standard left-invariant Levi-parallel contact Riemannian structure \((\eta, g)\) such that \(L_{\eta} = \Theta\) at the identity of \(M\).

In this case \(\xi\) is Killing, i.e. \(\tau = 0\) (Sasakian type).
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Let \((M_{n+1}(c), J, g)\) be a complex space form with \(c \neq 0\).

Let \(M \subset M_{n+1}(c)\) be a Levi non-degenerate real hypersurface.
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is an eigenvector of the shape operator \(A\).
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Assume \(M\) is a **Hopf hypersurface**, i.e. the tangent vector field

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is an eigenvector of the shape operator \(A\).
The 1-form \(\eta\) dual of \(\xi\) is a contact form.
The Riemannian metric \(g\) induced on \(M\) is an admissible metric.
Hopf hypersurfaces with constant principal curvatures:

- Takagi’s list of Hopf hypersurfaces in $\mathbb{C}P^n A_1, A_2, B, C, D, E$
- Berndt’s list of Hopf hypersurfaces in $\mathbb{C}H^n A_0, A_1, A_2, B$

In our case

If $M \subset \mathbb{C}P^n$ then $(M, \eta, g)$ is Levi-parallel if and only if $M$ is locally congruent to a hypersurface of type $A_1, A_2, B, C, D, E$

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Consequently:

$(M, \eta, g)$ is Levi-parallel $\iff$ the shape operator $A$ is $\eta$-parallel

Remark

Our condition of Levi-parallelism is equivalent to the one studied by Cho (2006), using a different linear connection.
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**Remark**

Our condition of Levi-parallelism is equivalent to the one studied by Cho (2006), using a different linear connection.
A basic formula

Let $(M, \eta, g)$ Levi-parallel contact Riemannian manifold. Let $l : TM \to TM$ be the Jacobi operator defined by

$$lX := R(X, \xi)\xi \quad X \in \mathcal{X}(M).$$

Then,

$$l = -\varphi^2 - \tau^2 - \nabla_\xi \tau.$$
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If \(\tau = 0\) (Sasakian type), then \(l = -\varphi^2\) and all the \(\xi\)-sectional curvatures are positive.

(the \(\xi\)-sectional curvatures are the sectional curvatures \(K(X, \xi)\) of the 2-planes containing the direction of \(\xi\))
Proposition (D-Lotta, 2011)

Let $(M, g)$ be a Riemannian manifold. The following are equivalent:

a) $M$ admits a contact form $\eta$ such that $(M, \eta, g)$ is a Levi-parallel contact Riemannian manifold of Sasakian type.

b) $M$ admits a global unit Killing vector field $\xi$ such that

i) $K(X, \xi) > 0$ for every $X \in [\xi]^{\perp}$

ii) $R(X, Y)\xi = 0$ for every $X, Y \in [\xi]^{\perp}$. 
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Remark

Sasakian manifolds \(\iff K(X, \xi) = 1\)
The following result is analogous to a result of Blair in contact metric geometry.

**Theorem (D-Lotta, 2011)**

Let $(M, \eta, g)$ be a Levi-parallel contact Riemannian manifold of dimension $\geq 5$. Assume that $R(X,Y)\xi = 0$ for any vector fields $X, Y$.

Then $S = \{-\lambda\}$, $\lambda > 0$, and $M$ is locally isometric to the Riemannian product $E_{n+1} \times S_n(4\lambda)$. 

Levi-parallel contact Riemannian manifolds Giulia Dileo University of Bari (Italy) Zlatibor, September 2012
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The Einstein case

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Let \((M, g, \eta)\) be a Levi-parallel contact Einstein Riemannian manifold of dimension \(\geq 5\). Assume \(\nabla_{\xi} l = 0\).
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Corollary

Let \((M, g, \eta)\) be a Levi-parallel contact Riemannian manifold of dimension \(\geq 5\). Assume that \(M\) has constant sectional curvature \(c\).

Then \(M\) is of Sasakian type, \(c > 0\) and \(S = \{-c\}\).
Steps of the proof

\[ \tilde{\nabla} \tilde{R}(X,Y,Z,W) = - \tilde{R}(Y,X,Z,W) = - \tilde{R}(X,Y,W,Z) = \tilde{R}(X,Y,Z,W) - \tilde{R}(Z,W,X,Y) = 2d\eta(Y,Z)g(\tau X,W) + 2d\eta(X,W)g(\tau Y,Z) - 2d\eta(X,Z)g(\tau Y,W) - 2d\eta(Y,W)g(\tau X,Z) + 2d\eta(Y,Z)g(\tau X,\phi W) + 2d\eta(X,\phi W)g(\tau Y,Z) - 2d\eta(X,Z)g(\tau Y,\phi W) - 2d\eta(Y,\phi W)g(\tau X,Z) \]
Steps of the proof

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- $\tilde{R}(X, Y, Z, W) - \tilde{R}(Z, W, X, Y) =$
  
  $= 2d\eta(Y, Z)g(\tau X, W) + 2d\eta(X, W)g(\tau Y, Z) - 2d\eta(X, Z)g(\tau Y, W) - 2d\eta(Y, W)g(\tau X, Z)$
Steps of the proof

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- $\tilde{\nabla}(X, Y, Z, W) = -\tilde{\nabla}(Y, X, Z, W) = -\tilde{\nabla}(X, Y, W, Z)$

- $\tilde{\nabla}(X, Y, Z, W) - \tilde{\nabla}(Z, W, X, Y) =$
  $$2d\eta(Y, Z)g(\tau X, W) + 2d\eta(X, W)g(\tau Y, Z)$$
  $$- 2d\eta(X, Z)g(\tau Y, W) - 2d\eta(Y, W)g(\tau X, Z)$$

- $\tilde{\nabla}(\varphi X, Y, Z, W) + \tilde{\nabla}(X, \varphi Y, Z, W) = 0$
Steps of the proof

I) Symmetries of the curvature tensor of $\tilde{\nabla}$


- $\tilde{R}(X, Y, Z, W) - \tilde{R}(Z, W, X, Y) =
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- $\tilde{R}(\varphi X, Y, Z, W) + \tilde{R}(X, \varphi Y, Z, W) = 0$

- $\tilde{R}(X, Y, \varphi Z, W) + \tilde{R}(X, Y, Z, \varphi W) =
  = 2d\eta(Y, \varphi Z)g(\tau X, W) + 2d\eta(X, W)g(\tau Y, \varphi Z) - 2d\eta(X, \varphi Z)g(\tau Y, W) - 2d\eta(Y, W)g(\tau X, \varphi Z) + 2d\eta(Y, Z)g(\tau X, \varphi W) + 2d\eta(X, \varphi W)g(\tau Y, Z) - 2d\eta(X, Z)g(\tau Y, \varphi W) - 2d\eta(Y, \varphi W)g(\tau X, Z)$
II) Interaction of the Ricci tensor $s$ of $\tilde{\nabla}$ with $\varphi$

\[
s(\varphi X, \varphi Y) = 2 \sum_{i=1}^{n} \tilde{R}(\varphi e_i, Y, \varphi e_i, X) + 4 g(\tau \varphi, X, Y) - 2 \text{tr}(\varphi^2) g(\tau \varphi, X, Y).
\]

For $Y \in D(\lambda)$ eigendistribution of $\varphi^2$, $\lambda \in S$

\[
\text{For } Y \in D(\lambda) \text{ eigendistribution of } \varphi^2, \lambda \in S \quad \varphi X, \varphi Y) = -s(X,Y) + 2(2\lambda - \text{tr}(\varphi^2)) g(\tau \varphi, X, Y).
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II) Interaction of the Ricci tensor $s$ of $\tilde{\nabla}$ with $\varphi$

Levi-parallelism $\sim s$ satisfies for $X, Y \in D$:

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$$s(\varphi X, \varphi Y) = -\lambda s(X, Y) + 2(2\lambda - tr(\varphi^2))g(\tau \varphi X, Y).$$
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$$s(X, Y) = Ric(X, Y) - 2g(\varphi^2 X, Y) + g((\nabla_\xi \tau) X, Y), \ X, Y \in D.$$
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\[
\begin{align*}
g \text{ Einstein metric} \\
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\end{align*}
\]

Taking $X \in D(\lambda), \ Y \in D$ and comparing with

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With $g$ Einstein metric

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III) Comparison of $s$ and $Ric$:

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and thus

$$\tau = 0.$$
Einstein examples with $\tau = 0$ and non trivial spectrum:
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$$(M_1, J_1, g_1), \ldots, (M_k, J_k, g_k) \text{ Kähler-Einstein with } c_1(M_i) > 0$$
Einstein examples with $\tau = 0$ and non trivial spectrum:

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$$c_1(M_i) = q_i \alpha_i, \quad q_i > 0, \quad \alpha_i \text{ indivisible in } H^2(M_i, \mathbb{Z})$$

Normalize $g_i$ in such a way that $\text{Ric}_{g_i} = q_i g_i$. 
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$$\pi : P \to B$$

be a non trivial principal $S^1$-bundle on $B := M_1 \times \cdots \times M_k$. 
Einstein examples with $\tau = 0$ and non trivial spectrum:

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Assume $P$ has non-trivial Euler class

\[e(P) = \sum b_i \pi_i^* \alpha_i, \quad b_i \in \mathbb{Z}, \quad \pi_i : B \to M_i.\]
Einstein examples with $\tau = 0$ and non trivial spectrum:

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Assume $P$ has non-trivial Euler class

$$e(P) = \sum b_i \pi_i^* \alpha_i, \quad b_i \in \mathbb{Z}, \quad \pi_i : B \to M_i.$$  

Denote by $\eta$ be the connection form on $P$ such that

$$d\eta = \pi^* \Omega, \quad \Omega := \sum b_i \pi_i^* \omega_i,$$

where $\omega_i$ is the Kähler form of $M_i$. 
According to a result of Wang-Ziller (1990), up to scaling there exists a unique Einstein metric $g$ on $P$ such that

$$\pi : (P, g) \to (B, g_0)$$

is a Riemannian submersion with totally geodesic fibers and

$$g_0 = x_1 g_1 \perp \cdots \perp x_k g_k, \quad x_i > 0.$$
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We choose the scaling factor in such a way that

$$g = \pi^* g_o + \eta \otimes \eta.$$
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We choose the scaling factor in such a way that

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g = \pi^* g_0 + \eta \otimes \eta.
$$

Then $(P, \eta, g)$ is a Levi-parallel contact Riemannian manifold of Sasakian type.

The spectrum $S$ depends on the constants $x_1, \ldots, x_k, b_1, \ldots, b_k$. 
Classification of locally symmetric metrics

Theorem (D-Lotta, 2011)

Let $(M, \eta, g)$ be a Levi parallel contact locally symmetric Riemannian manifold of dimension $2n + 1 \geq 5$.

Moreover, either $M$ is of Sasakian type with constant sectional curvature $\lambda$, or $M$ is locally isometric to the Riemannian product $E_n + 1 \times S_n(4\lambda)$. 

Levi-parallel contact Riemannian manifolds Giulia Dileo University of Bari (Italy) Zlatibor, September 2012
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- \(M\) is of Sasakian type with constant sectional curvature \(\lambda\), or
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Core of the proof: $\mathcal{S} = \{-\lambda\}$. 

Levi-parallel contact Riemannian manifolds  Giulia Dileo  University of Bari (Italy)  Zlatibor, September 2012
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computations involving $\nabla$, $\tilde{\nabla}$, curvature and Ricci tensors

In both cases the spectrum is trivial.
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There are two possibilities for the Jacobi operator $l$:
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There are two possibilities for the Jacobi operator $l$:

- $l = 0$
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There are two possibilities for the Jacobi operator $l$:

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- $l = c\,Id$ on $D$, $c \neq 0$
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There are two possibilities for the Jacobi operator $l$:

- $l = 0 \Rightarrow R(X, Y)\xi = 0$ for every vector fields $X, Y$
- $l = c \text{Id on } D$, $c \neq 0$
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Taking the homothetic deformation

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\varphi' = \frac{1}{\sqrt{\lambda}} \varphi, \quad \xi' = \frac{1}{\sqrt{\lambda}} \xi, \quad \eta' = \sqrt{\lambda} \eta, \quad g' = \lambda g,
\]

we get a (CR integrable) contact metric structure.
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The result follows from the Boeckx-Cho classification of locally symmetric contact metric manifolds.


