Totally geodesic subalgebras for various inner products on nilpotent Lie algebras

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Table of contents

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Then, we have the *descending central series* of $\mathfrak{g}$:

$$\mathfrak{g} = \mathcal{C}^0(\mathfrak{g}) \supset \mathcal{C}^1(\mathfrak{g}) \supset \cdots \supset \mathcal{C}^k(\mathfrak{g}) \supset \ldots .$$

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Definition

A Lie algebra $\mathfrak{g}$ is called **nilpotent** if there is an integer k such that

$$\mathcal{C}^k(\mathfrak{g}) = \{0\}.$$

The smallest integer k such that $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ is called **the nilindex** of $\mathfrak{g}$.

Examples of nilpotent Lie algebras

1. Every **abelian** Lie algebra is nilpotent with the nilindex equal to 1.
2. The **Heisenberg algebra** $\mathfrak{h}_{2k+1}$ defined in the basis $\{X_1, X_2, \ldots, X_{2k+1}\}$ by

$$[X_{2i-1}, X_{2i}] = X_{2k+1} \ , \quad i = 1, \ldots, k.$$

   The nilindex is equal to 2.
3. The n-dimensional algebra $\mathfrak{m}_0(n)$ defined in a basis $\{X_1, \ldots, X_n\}$ by the brackets

$$[X_1, X_i] = X_{i+1} \text{ for all } 2 \leq i \leq n-1.$$

   The nilindex is equal to $n-1$.

Metric Lie algebras

⋆ Let $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ be a Lie algebra equipped with an inner product (a **metric Lie algebra**).

⋆ (G, g)-the corresponding simply connected Lie group with left-invariant Riemannian metric g.

⋆ If $\mathfrak{g}$ is a metric Lie algebra, with inner product $\langle\cdot,\cdot\rangle$, the **Levi-Civita connection** on $\mathfrak{g}$ is given by:

$$2\langle\nabla_X Y, Z\rangle = \langle[X,Y],Z\rangle + \langle[Z,X],Y\rangle + \langle[Z,Y],X\rangle, \quad \forall X,Y,Z \in \mathfrak{g}. \quad (1)$$

Totally geodesic subalgebras

Definition (Totally geodesic submanifolds)

For a Riemannian manifold (M, g), a submanifold M' is said to be **totally geodesic** if, for any vector fields $X, Y \in T(M')$, $\nabla_X Y$ is in $T(M')$.

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ and consider the corresponding connected subgroup H of G.

$\star$ A subalgebra $\mathfrak{h}$ of a metric Lie algebra $\mathfrak{g}$ is said to be **totally geodesic subalgebra** if the Lie subgroup H corresponding to $\mathfrak{h}$ is a totally geodesic submanifold relative to the left-invariant Riemannian metric defined by the inner product, on the simply connected Lie group G associated to $\mathfrak{g}$.

Lemma

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ and let $\mathfrak{h}^\perp$ denote the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$.
Then $\mathfrak{h}$ is a **totally geodesic subalgebra** of $\mathfrak{g}$ if and only if

$$\langle [Z, X], Y\rangle + \langle [Z, Y], X\rangle = 0, \text{ for all } Z \in \mathfrak{h}^\perp, X, Y \in \mathfrak{h}. \tag{2}$$

Totally geodesic subalgebras of nilpotent Lie algebras

Proposition 1.

1. Every metric Lie algebra possesses a geodesic.
2. If $\mathfrak{g}$ is a nilpotent Lie algebra and $Y \in \mathfrak{g}$ is nonzero, then there is an inner product on $\mathfrak{g}$ for which Y is a geodesic.

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Proposition 2.

If $\mathfrak{g}$ be a nilpotent metric Lie algebra and $\mathfrak{h}$ is a totally geodesic subalgebra of $\mathfrak{g}$ of **codimension one**, then $\mathfrak{g}$ is a direct sum of Lie ideals,

$$\mathfrak{g} \cong \mathfrak{h} \oplus \mathbb{R}.$$

Totally geodesic subalgebras of two-step nilpotent Lie algebras

Define the linear map

$$j : \mathfrak{z}(\mathfrak{g}) \to \mathfrak{so}(\mathfrak{z}^{\perp})$$
$$\langle [X, Y], Z \rangle = \langle j(Z)X, Y \rangle \quad \text{for} \quad X, Y \in \mathfrak{z}^{\perp}(\mathfrak{g}), Z \in \mathfrak{z}(\mathfrak{g}). \tag{3}$$

Nonsingular two-step nilpotent Lie algebras $\mathfrak{g}$ are defined by the condition that for all $X \notin \mathfrak{z}(\mathfrak{g})$ the map $\text{ad}(X) : \mathfrak{g} \to \mathfrak{z}(\mathfrak{g})$ is surjective.

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Theorem (P. Eberlein, 1994)

Let $\mathfrak{g}$ be a nonsingular metric two-step nilpotent Lie algebra. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is totally geodesic if and only if exactly of one the following occurs:

(a) $\mathfrak{h}$ is a subspace of $\mathfrak{z}^{\perp}(\mathfrak{g})$;
(b) $\mathfrak{h}$ is a subspace of $\mathfrak{z}(\mathfrak{g})$;
(c) $\mathfrak{h}$ is a nontrivial direct sum $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{z}(\mathfrak{g})) \oplus (\mathfrak{h} \cap \mathfrak{z}^{\perp}(\mathfrak{g}))$ and $j(Z)$ leaves invariant $\mathfrak{h} \cap \mathfrak{z}^{\perp}(\mathfrak{g})$ for all $Z \in \mathfrak{h} \cap \mathfrak{z}(\mathfrak{g})$.

Filiform Lie algebras

Definition (Filiform Lie algebras)

A nilpotent Lie algebra $\mathfrak{g}$ of dimension n is said to be **filiform** if it possesses an element of maximal nilpotency; that is,

$$\text{there exists } X \in \mathfrak{g} \text{ with } \operatorname{ad}^{n-2}(X) \neq 0, \tag{4}$$

where $\operatorname{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is the adjoint map $\operatorname{ad}(X)(Y) = [X, Y]$.

The Kerr-Payne result

Consider a standard metric filiform Lie algebra $\mathfrak{n}$ defined in an orthonormal basis $\{X_1, X_2, \ldots, X_n\}$ such that

$$[X_1, X_i] = c_i X_{i+1}, \quad \text{for all } i = 2, \ldots, n-1,$$

where constants $c_i \neq 0$.

Theorem (M. Kerr and T.Payne, 2010)

If $\mathfrak{h}$ is a totally geodesic proper subalgebra of the **filiform metric Lie algebra** $\mathfrak{n}$, then

$$\dim(\mathfrak{h}) \leq \dim(\mathfrak{n})/2.$$

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⋆ Moreover, they obtained that $\mathfrak{h}$ is a proper totally geodesic subalgebra ($\dim(\mathfrak{h}) \geq 2$) of standard metric filiform Lie algebra $\mathfrak{n}$ if and only if it is a subspace of a certain cone which is a codimension one subset of $X_1^\perp$.

Filiform Lie algebras

Theorem (c.f. Michèle Vergne, 1970)

Every filiform nilpotent Lie algebra has a basis $\{X_1, \ldots, X_n\}$ such that

$$[X_1, X_i] = X_{i+1}, \text{ for all } i \geq 2,$$

$$[X_i, X_j] \in \mathfrak{g}_{i+j}, \text{ for all } i, j \text{ with } i + j \neq n + 1,$$

$$\exists \alpha \in \mathbb{R} \text{ such that } [X_i, X_{n-i+1}] = (-1)^i \alpha X_n, \text{ for all } 2 \leq i \leq n - 1, \quad (5)$$

$$\text{if } n \text{ is odd, } \alpha = 0,$$

where $\mathfrak{g}_k = \text{Span}\{X_k, \ldots, X_n\}$ and for convenience we have set $X_i = 0$ for $i > n$.

$\star$ We will say that a filiform nilpotent Lie algebra $\mathfrak{g}$ is **regular** if $\mathfrak{g}$ has a basis $\{X_1, \ldots, X_n\}$ satisfying the conditions of Theorem, with $\alpha = 0$. Otherwise, $\mathfrak{g}$ will be called **irregular**.

Totally geodesic subalgebras of filiform Lie algebras

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⋆ Suppose that $\mathfrak{g}$ is a filiform nilpotent metric Lie algebra, $\mathfrak{h}$ is a proper subalgebra of $\mathfrak{g}$ and **$\mathfrak{h}^\perp$ is $\mathfrak{h}$-invariant** i.e. $[\mathfrak{h}^\perp, \mathfrak{h}] \subseteq \mathfrak{h}^\perp$. Then

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$$\dim(\mathfrak{h}) \leq \dim(\mathfrak{g})/2.$$

⋆ Consider the following 6−dimensional filiform Lie algebra $\mathfrak{g}$

$$[X_1, X_i] = X_{i+1}, \quad \text{for } i = 2, \dots, 5,$$
$$[X_2, X_3] = -X_6.$$

For no choice of inner product, does $\mathfrak{g}$ possess a totally geodesic subalgebra of dimension > 2.

$\mathbb{N}$-graded filiform nilpotent Lie algebras

Definition ($\mathbb{N}$-graded filiform Lie algebras)

An n-dimensional nilpotent Lie algebra $\mathfrak{g}$ is **$\mathbb{N}$-graded filiform**, if it can be decomposed in a direct sum of one dimensional subspaces $\mathfrak{g} = \dotplus_{i=1}^{n} V_i$ with

$$\begin{aligned} [V_1, V_i] &= V_{i+1}, \quad \text{for all } i > 1 \text{ and} \\ [V_i, V_j] &\subseteq V_{i+j}, \quad \text{for all } i, j \in \mathbb{N} \end{aligned} \tag{6}$$

where for convenience we set $V_i = 0$ for $i > n$.

$\mathbb{N}$-graded filiform Lie algebras

Theorem (D. Millionshchikov, 2004)

Let $\mathfrak{g} = \bigoplus_{i=1}^{n} V_i$ be an $\mathbb{N}$-**graded filiform Lie algebra**.

Then $\mathfrak{g}$ is isomorphic to the one and only one Lie algebra from the following list:

1. Lie algebras of the six sequences $\mathfrak{m}_0(n)$, $\mathfrak{m}_2(n)$, $\mathcal{V}_n$, $\mathfrak{m}_{0,1}(2k+1)$, $\mathfrak{m}_{0,2}(2k+2)$, $\mathfrak{m}_{0,3}(2k+3)$, defined by the basis $X_1, \dots, X_n$ and commutating relations in Table 1
2. Lie algebras of 5 one-parameter families $\mathfrak{g}_{n,\alpha}$ of dimensions $n = 7, \dots, 11$ respectively, defined by their basses and Lie structure relations in Table 2

algebra	dimension	presentation
$\mathfrak{m}_0(n)$	$n \geq 3$	$[X_1, X_i] = X_{i+1}, \quad i = 2, \dots, n-1$
$\mathfrak{m}_2(n)$	$n \geq 5$	$[X_1, X_i] = X_{i+1}, \quad i = 2, \dots, n-1$ $[X_2, X_i] = X_{i+2}, \quad i = 3, \dots, n-2$
$\mathcal{V}_n$	$n \geq 12$	$[X_i, X_j] = \begin{cases} (j-i)X_{i+j}, & i+j \leq n; \\ 0, & i+j > n; \end{cases}$
$\mathfrak{m}_{0,1}(2k+1)$, $k \geq 3$	$n = 2k+1$	$[X_1, X_i] = X_{i+1}, \quad i = 2, \dots, 2k$ $[X_l, X_{2k-l+1}] = (-1)^{l+1} X_{2k+1}, \quad l = 2, \dots, k.$
$\mathfrak{m}_{0,2}(2k+2)$, $k \geq 3$	$n = 2k+2$	$[X_1, X_i] = X_{i+1}, \quad i = 2, \dots, 2k+1$ $[X_l, X_{2k-l+1}] = (-1)^{l+1} X_{2k+1}, \quad l = 2, \dots, k$ $[X_j, X_{2k-j+2}] = (-1)^{j+1}(k-j+1) X_{2k+2}, \quad j = 2, \dots, k$
$\mathfrak{m}_{0,3}(2k+3)$, $k \geq 3$	$n = 2k+3$	$[X_1, X_i] = X_{i+1}, \quad i = 2, \dots, 2k+2$ $[X_l, X_{2k-l+1}] = (-1)^{l+1} X_{2k+1}, \quad l = 2, \dots, k$ $[X_j, X_{2k-j+2}] = (-1)^{j+1}(k-j+1) X_{2k+2}, \quad j = 2, \dots, k$ $[X_m, X_{2k-m+3}] = (-1)^m \left((m-2)k - \frac{(m-2)(m-1)}{2}\right) X_{2k+3},$ $m = 3, \dots, k+1$

$\star$ $\mathfrak{m}_0(3) \cong \mathfrak{m}_2(3) \cong \mathcal{V}_3$, $\mathfrak{m}_0(4) \cong \mathfrak{m}_2(4) \cong \mathcal{V}_4$, $\mathfrak{m}_2(5) \cong \mathcal{V}_5$, $\mathfrak{m}_2(6) \cong \mathcal{V}_6$.

algebra	restrictions	presentation
$\mathfrak{g}_{7,\alpha}$	$\alpha \neq -2$	$[X_1, X_j] = X_{j+1}, \quad 2 \leq j \leq 6$ $[X_2, X_3] = (2+\alpha)X_5, \quad [X_2, X_4] = (2+\alpha)X_6,$ $[X_2, X_5] = (1+\alpha)X_7, \quad [X_3, X_4] = X_7,$
$\mathfrak{g}_{8,\alpha}$	$\alpha \neq -2$	relations of $\mathfrak{g}_{7,\alpha}$ and: $[X_1, X_7] = X_8, \quad [X_2, X_6] = \alpha X_8, \quad [X_3, X_5] = X_8,$
$\mathfrak{g}_{9,\alpha}$	$\alpha \neq -\frac{5}{2}, -2$	relations of $\mathfrak{g}_{8,\alpha}$ and: $[X_1, X_8] = X_9, \quad [X_2, X_7] = \frac{2\alpha^2+3\alpha-2}{2\alpha+5}X_9,$ $[X_3, X_6] = \frac{2\alpha+2}{2\alpha+5}X_9, \quad [X_4, X_5] = \frac{3}{2\alpha+5}X_9,$
$\mathfrak{g}_{10,\alpha}$	$\alpha \neq -\frac{5}{2}$	relations of $\mathfrak{g}_{9,\alpha}$ and: $[X_1, X_9] = X_{10}, \quad [X_2, X_8] = \frac{2\alpha^2+\alpha-1}{2\alpha+5}X_{10},$ $[X_3, X_7] = \frac{2\alpha-1}{2\alpha+5}X_{10}, \quad [X_4, X_6] = \frac{3}{2\alpha+5}X_{10},$
$\mathfrak{g}_{11,\alpha}$	$\alpha \neq -\frac{5}{2}, -1, -3$	relations of $\mathfrak{g}_{10,\alpha}$ and: $[X_1, X_{10}] = X_{11}, \quad [X_2, X_9] = \frac{2\alpha^3+2\alpha^2+3}{2(\alpha^2+4\alpha+3)}X_{11},$ $[X_3, X_8] = \frac{4\alpha^3+8\alpha^2-8\alpha-21}{2(\alpha^2+4\alpha+3)(2\alpha+5)}X_{11}, \quad [X_4, X_7] = \frac{3(2\alpha^2+4\alpha+5)}{2(\alpha^2+4\alpha+3)(2\alpha+5)}$ $[X_5, X_6] = \frac{3(4\alpha+1)}{2(\alpha^2+4\alpha+3)(2\alpha+5)}X_{11}$

⋆ $\mathfrak{g}_{n,8} \cong \mathcal{V}_n$ where $n = 7, \dots, 11$ (the relevant basis $\{X_1, \frac{1}{(k-2)! \cdot 60}X_k : k = 2, \dots, n\}$)

⋆ $\mathfrak{g}_{7,-2} \cong \mathfrak{m}_{0,1}(7)$, $\mathfrak{g}_{8,-2} \cong \mathfrak{m}_{0,2}(8)$ and $\mathfrak{g}_{9,-2} \cong \mathfrak{m}_{0,3}(9)$.

Totally geodesic subalgebras of $\mathbb{N}$-graded filiform Lie algebras

⋆ Take an inner product on an $\mathbb{N}$-graded filiform Lie algebra $\mathfrak{g}$ for which $X_1, \dots, X_n$ are orthonormal.
Then the subalgebra

$$\mathfrak{h} = \mathrm{Span}(X_i : i \text{ is even})$$

is a totally geodesic subalgebra of dimension $\lfloor \frac{\dim(\mathfrak{g})}{2} \rfloor$.

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Theorem (G. Cairns, A. Hinić Galić, Y. Nikolayevsky)

The dimension of a totally geodesic subalgebra $\mathfrak{h}$ of an n-dimensional metric $\mathbb{N}$-graded filiform Lie algebras $\mathfrak{g}$ is

$$\leq \lfloor \frac{\dim(\mathfrak{g})}{2} \rfloor,$$

except if

1. $\mathfrak{g} \cong \mathfrak{m}_0(n)$ when $\dim(\mathfrak{h}) \leq \dim(\mathfrak{g}) - 2$, or
2. $\mathfrak{g} \cong \mathfrak{m}_{0,1}(2k+1)$ when $\dim(\mathfrak{h}) \leq \dim(\mathfrak{g}) - 4$.

Case $\mathfrak{m}_0(n)$: Define an orthonormal basis $\{E_1, \ldots, E_n\}$ as follows: set $E_1 = X_1, E_n = X_n$ and for each $i \in \{2, 3, \ldots, n-1\}$, pose

$$E_i = \sum_{j=0}^{\lfloor \frac{n-1-i}{2} \rfloor} \binom{n-1-i-j}{j} X_{i+2j}. \tag{7}$$

The basis has been chosen so that

$$[E_1, E_i] = E_{i+1} + E_{i+3} + E_{i+5} + \ldots, \quad \text{for all } i \geq 2. \tag{8}$$

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Then, totally geodesic subalgebra $\mathfrak{h}$ of codimension two is spanned by the vectors $Y_2, Y_3, \dots, Y_{n-2}, Y_n$, where for $i \in \{2, 3, \dots, n-2, n\}$,

$$Y_i = \begin{cases} E_i & : \text{ if } n-i \text{ is even,} \\ E_i - E_{n-1} & : \text{ otherwise.} \end{cases} \tag{9}$$

The orthogonal complement $\mathfrak{h}^{\perp}$ to $\mathfrak{h}$ is spanned by the vectors

$$Z_1 := E_1 \text{ and } Z_2 := \sum_{\substack{1 \leq j \leq n-2 \\ j \text{ odd}}} E_{n-j}.$$

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⋆ If a filiform nilpotent metric Lie algebra $\mathfrak{g}$ of dimension n possesses a totally geodesic subalgebra of codimension two, then $\mathfrak{g}$ is isomorphic to the filiform Lie algebra $\mathfrak{m}_0(n)$.

Case $\mathfrak{m}_{0,1}(2k+1)$: Let $\mathfrak{m} = \mathbb{R}^{2k+1}$, $k \geq 3$, be a Euclidean space with the inner product $\langle \cdot, \cdot \rangle$ and an orthonormal basis E_i, $i = 1, \dots, 2k+1$.

Introduce the subspace $\mathfrak{m}' = \mathrm{Span}(E_2, \dots, E_{2k+1})$, and define a bilinear skew-symmetric map $[\cdot,\cdot] : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ by

$$[E_1, X] = NX, \quad [X, Y] = \langle KX, Y \rangle E_{2k+1}, \quad \text{for all } X, Y \in \mathfrak{m}', \tag{10}$$

where $N, K \in \mathrm{End}(\mathfrak{m}')$ are defined by their matrices relative to the orthonormal basis $E_2, \dots, E_{2k+1}$ for $\mathfrak{m}'$ as follows:

$$N = \left(\begin{array}{cc|cc} 0 & S & 0 & 0 \\ -S + uu^t & 0 & 0 & 0 \\ \hline p^t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right), \qquad K = \left(\begin{array}{cc|cc} 0 & I_{k-1} & 0 & 0 \\ -I_{k-1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \tag{11}$$

where I_{k-1} is the identity matrix, $u, p \in \mathbb{R}^{k-1}$, and S is a symmetric nonsingular $(k-1) \times (k-1)$-matrix such that the matrix $T = S(-S + uu^t)$ is nilpotent.

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$\star$ The subalgebra $\mathfrak{h} = (\text{Span}(E_1, E_{2k}, (0, u, 0_{k+1})^t, (0_k, u, 0, 0)^t))^\perp$ is a totally geodesic subalgebra of $\mathfrak{m}$ of codimension 4.

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Introduce the subspace $\mathfrak{m}' = \mathrm{Span}(E_2, \ldots, E_{2k+1})$, and define a bilinear skew-symmetric map $[\cdot, \cdot] : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ by

$$[E_1, X] = NX, \quad [X, Y] = \langle KX, Y \rangle E_{2k+1}, \quad \text{for all } X, Y \in \mathfrak{m}', \tag{10}$$

where $N, K \in \mathrm{End}(\mathfrak{m}')$ are defined by their matrices relative to the orthonormal basis $E_2, \ldots, E_{2k+1}$ for $\mathfrak{m}'$ as follows:

$$N = \left(\begin{array}{cc|cc} 0 & S & 0 & 0 \\ -S + uu^t & 0 & 0 & 0 \\ \hline p^t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right), \qquad K = \left(\begin{array}{cc|cc} 0 & I_{k-1} & 0 & 0 \\ -I_{k-1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \tag{11}$$

where I_{k-1} is the identity matrix, $u, p \in \mathbb{R}^{k-1}$, and S is a symmetric nonsingular $(k-1) \times (k-1)$-matrix such that the matrix $T = S(-S + uu^t)$ is nilpotent.

⋆ The subalgebra $\mathfrak{h} = (\mathrm{Span}(E_1, E_{2k}, (0, u, 0_{k+1})^t, (0_k, u, 0, 0)^t))^{\perp}$ is a totally geodesic subalgebra of $\mathfrak{m}$ of codimension 4.

⋆ It can be shown that for a suitable choice of S, u and p, $\mathfrak{m} \cong \mathfrak{m}_{0,1}(2k+1)$.

THANK YOU FOR YOUR ATTENTION