ON QUASI-EINSTEIN MANIFOLDS

Ryszard Deszcz and Małgorzata Głogowska

XVII GEOMETRICAL SEMINAR
3 - 8 September, 2012
Zlatibor, Serbia

Ryszard Deszcz and Małgorzata Głogowska
Department of Mathematics
Wrocław University of Environmental and Life Sciences, Poland

Contents

Some endomorphisms

Let (M, g) be a connected n-dimensional, $n = \dim M \geq 3$, semi-Riemannian manifold of class C^∞ and ∇ its Levi-Civita connection. We define on M the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X, Y)$ and $\mathcal{C}(X, Y)$ by

$$
\begin{aligned}
(X \wedge_A Y)Z &= A(Y, Z)X - A(X, Z)Y, \\
\mathcal{R}(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \\
\mathcal{C}(X, Y) &= \mathcal{R}(X, Y) - \frac{1}{n-2}(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y) \\
&\quad - \frac{\kappa}{(n-2)(n-1)} X \wedge_g Y,
\end{aligned}
$$

A - a symmetric $(0, 2)$-tensor,
S - the Ricci tensor, $\mathcal{S}$ - the Ricci operator, $g(\mathcal{S}X, Y) = S(X, Y)$,
κ - the scalar curvature,
$\Xi(M)$ - the Lie algebra of vector fields of M, $X, Y, Z \in \Xi(M)$.

Some $(0, 4)$-tensors

The Riemann-Christoffel curvature tensor R,
the Weyl conformal curvature tensor C
and the $(0, 4)$-tensor G of (M, g)
are defined by

$$R(X_1, X_2, X_3, X_4) = g(\mathcal{R}(X_1, X_2)X_3, X_4),$$

$$C(X_1, X_2, X_3, X_4) = g(\mathcal{C}(X_1, X_2)X_3, X_4),$$

$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge_g X_2)X_3, X_4),$$

respectively, where $X_1, \ldots, X_4 \in \Xi(M)$.

The Kulkarni-Nomizu product $E \wedge F$

For symmetric $(0,2)$-tensors E and F
we define their *Kulkarni-Nomizu product* $E \wedge F$ by

$$\begin{aligned}(E \wedge F)(X_1, X_2, X_3, X_4) &= E(X_1, X_4)F(X_2, X_3) + E(X_2, X_3)F(X_1, X_4) \\ &\quad - E(X_1, X_3)F(X_2, X_4) - E(X_2, X_4)F(X_1, X_3)\,,\end{aligned}$$

where $X_1, \ldots, X_4 \in \Xi(M)$.
Now the Weyl tensor C can be presented in the form

$$C = R - \frac{1}{n-2}\, g \wedge S + \frac{\kappa}{(n-2)(n-1)}\, G,$$

where $G = \frac{1}{2}\, g \wedge g$.

Some $(0,6)$-tensors

For a symmetric $(0,2)$-tensor A and a $(0,k)$-tensor T, $k \geq 1$, we define the $(0,k+2)$-tensors $R \cdot T$, $C \cdot T$ and $Q(A,T)$ by

$$(R \cdot T)(X_1, \ldots, X_k; X, Y) \ = \ (\mathcal{R}(X,Y) \cdot T)(X_1, \ldots, X_k)$$
$$= \ -T(\mathcal{R}(X,Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots, X_{k-1}, \mathcal{R}(X,Y)X_k),$$

$$(C \cdot T)(X_1, \ldots, X_k; X, Y) \ = \ (\mathcal{C}(X,Y) \cdot T)(X_1, \ldots, X_k)$$
$$= \ -T(\mathcal{C}(X,Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots, X_{k-1}, \mathcal{C}(X,Y)X_k),$$

$$Q(A,T)(X_1, \ldots, X_k; X, Y) \ = \ ((X \wedge_A Y) \cdot T)(X_1, \ldots, X_k)$$
$$= \ -T((X \wedge_A Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k),$$

respectively. Setting in the above formulas
$T = R$, $T = S$, $T = C$, $A = g$ or $A = S$
we obtain the tensors: $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$ and $R \cdot S$,
and $Q(g,R)$, $Q(g,C)$, $Q(S,R)$, $Q(S,C)$ and $Q(g,S)$.

Tachibana tensors

Let A be a symmetric $(0, 2)$-tensor and T a $(0, k)$-tensor.
The tensor $Q(A, T)$ is called the *Tachibana tensor of A and T*,
or the *Tachibana tensor* for short ([DGPSS]).

We like to point out that in some papers,
$Q(g, R)$ is called the *Tachibana tensor* (see e.g. [HV], [JHSV], [JHP-TV]).

[DGPSS] R. Deszcz, M. Głogowska, M. Plaue, K. Sawicz, and M. Scherfner, On hypersurfaces in space forms satisfying particular curvature conditions of Tachibana type, Kragujevac J. Math. 35 (2011), 223-247.

[HV] S. Haesen and L. Verstraelen, Properties of a scalar curvature invariant depending on two planes, Manuscripta Math. 122 (2007), 59–72.

[JHSV] B. Jahanara, S. Haesen, Z. Senturk and L. Verstraelen, On the parallel transport of the Ricci curvatures, J. Geom. Phys. 57 (2007), 1771–1777.

[JHP-TV] B. Jahanara, S. Haesen, M. Petrovic-Torgasev and L. Verstraelen, On the Weyl curvature of Deszcz, Publ. Math. Debrecen 74 (2009), 417–431.

Some subsets of semi-Riemannian manifolds

Let (M, g), $n \geq 4$, be a semi-Riemannian manifold.
We define the following subset of M:

$$\mathcal{U}_R = \{x \in M \,|\, R \neq \tfrac{\kappa}{(n-1)n}\, G \text{ at } x\},$$

$$\mathcal{U}_S = \{x \in M \,|\, S \neq \tfrac{\kappa}{n}\, g \text{ at } x\},$$

$$\mathcal{U}_C = \{x \in M \,|\, C \neq 0 \text{ at } x\}.$$

We can easily check that

$$\mathcal{U}_S \subset \mathcal{U}_R \text{ and } \mathcal{U}_C \subset \mathcal{U}_R.$$

Pseudosymmetric manifolds

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be *pseudosymmetric* if at every point of M the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

The manifold (M, g) is pseudosymmetric if and only if

$$R \cdot R \;=\; L_R \, Q(g, R)$$

holds on $\mathcal{U}_R$, where L_R is some function on this set.

Every *semisymmetric* manifold $(R \cdot R = 0)$ is pseudosymmetric.
The converse statement is not true.

Pseudosymmetric manifolds of constant type

According to [BKV], a pseudosymmetric manifold (M, g), $n \geq 3$, $(R \cdot R = L_R\, Q(g, R))$ is said to be *pseudosymmetric space of constant type* if the function L_R is constant on $\mathcal{U}_R \subset M$.

Theorem (cf. [D]). Every type number two hypersurface M isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 3$, is a pseudosymmetric space of constant type. Precisely,

$$R \cdot R \;=\; \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, R),$$

holds on $\mathcal{U}_R \subset M$, where $\widetilde{\kappa}$ is the scalar curvature of the ambient space.

[BKV] E. Boeckx, O. Kowalski, L. Vanhecke, Riemannian manifolds of Conullity Two, World Sci., Singapore.

[D] F. Defever, R. Deszcz, P. Dhooghe, L. Verstraelen and S. Yaprak, On Ricci-pseudo -symmetric hypersurfaces in spaces of constant curvature, Results in Math. 27 (1995), 227–236.

Ricci-pseudosymmetric manifolds

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be *Ricci-pseudosymmetric* if at every point of M the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent.

The manifold (M, g) is Ricci-pseudosymmetric if and only if

$$R \cdot S \;=\; L_S \, Q(g, S)$$

holds on $\mathcal{U}_S$, where L_S is some function on this set.

Every *Ricci-semisymmetric* manifold $(R \cdot S = 0)$ is Ricci-pseudosymmetric. The converse statement is not true.

Ricci-pseudosymmetric manifolds of constant type (1)

According to [G], a Ricci-pseudosymmetric manifold (M, g), $n \geq 3$, $(R \cdot S = L_S\, Q(g, S))$ is said to be
Ricci-pseudosymmetric manifold of constant type
if the function L_S is constant on $\mathcal{U}_S \subset M$.

[G] M. Glogowska, Curvature conditions on hypersurfaces with two distinct principal curvatures, in: Banach Center Publ. 69, Inst. Math. Polish Acad. Sci., 2005, 133–143.

Ricci-pseudosymmetric manifolds of constant type (2)

Theorem (cf. [DY]). If M is a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 3$, such that at every point of M there are principal curvatures $0, \ldots, 0, \lambda, \ldots, \lambda, -\lambda, \ldots, -\lambda$, with the same multiplicity of λ and $-\lambda$, and λ is a positive function on M, then M is a Ricci-pseudosymmetric manifold of constant type. Precisely,

$$R \cdot S \;=\; \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, S)$$

holds on M. In particular, every Cartan hypersurface is a Ricci-pseudosymmetric manifold of constant type.

[DY] R. Deszcz and S. Yaprak, Curvature properties of Cartan hypersurfaces, Colloq. Math. 67 (1994), 91–98.

Weyl-pseudosymmetric manifolds (1)

A semi-Riemannian manifold (M, g), $n \geq 4$, is said to be *Weyl-pseudosymmetric* if at every point of M the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent.

The manifold (M, g) is Weyl-pseudosymmetric if and only if

$$R \cdot C \ = \ L_C \, Q(g, C)$$

holds on $\mathcal{U}_C$, where L_C is some function on this set.

Weyl-pseudosymmetric manifolds (2)

Every pseudosymmetric manifold ($R \cdot R = L_R\, Q(g, R)$)
is Weyl-pseudosymmetric ($R \cdot C = L_R\, Q(g, C)$).
In particular, every semisymmetric manifold ($R \cdot R = 0$)
is Weyl-semisymmetric ($R \cdot C = 0$).

If $\dim M \geq 5$ the converse statement are true. Precisely, if
$R \cdot C = L_C\, Q(g, C)$, resp. $R \cdot C = 0$, is satisfied on $\mathcal{U}_C \subset M$, then
$R \cdot R = L_C\, Q(g, R)$, resp. $R \cdot R = 0$, holds on $\mathcal{U}_C$.

Weyl-pseudosymmetric manifolds (3)

An example of a 4-dimensional Riemannian manifold satisfying $R \cdot C = 0$ with non-zero tensor $R \cdot R$ was found by A. Derdziński ([D]).

An example of a 4-dimensional submanifold in a 6-dimensional Euclidean space $\mathbb{E}^6$ satisfying $R \cdot C = 0$ with non-zero tensor $R \cdot R$ was found by G. Zafindratafa ([Z]).

[D] A. Derdziński, Exemples de metriques de Kaehler et d'Einstein autoduales sur le plan complexe, in: Geometrie riemannianne en dimension 4 (Seminaire Arthur Besse 1978/79), Cedic/Fernand Nathan, Paris 1981, 334–346.

[Z] G.K. Zafindratafa, Sous - varietes soumieses a des conditions de courbure, These de doctorat, Faculteit Wetenschappen, Katholieke Universiteit Leuven, 1991.

Relations between some classes of manifolds (1)

Inclusions between mentioned classes of manifolds can be presented in the following diagram ([DGHS]).
We mention that all inclusions are strict, provided that $n \geq 4$.

[DGHS] R. Deszcz, M. Głogowska, M. Hotlos, and K. Sawicz, A Survey on Generalized Einstein Metric Conditions, Advances in Lorentzian Geometry: Proceedings of the Lorentzian Geometry Conference in Berlin, AMS/IP Studies in Advanced Mathematics 49, S.-T. Yau (series ed.), M. Plaue, A.D. Rendall and M. Scherfner (eds.), 2011, 27–46.

Relations between some classes of manifolds (2)

$R \cdot S = L_S Q(g, S)$	$\supset$	$R \cdot R = L_R Q(g, R)$	$\subset$	$R \cdot C = L_C Q(g, C)$
$\cup$		$\cup$		$\cup$
$R \cdot S = 0$	$\supset$	$R \cdot R = 0$	$\subset$	$R \cdot C = 0$
$\cup$		$\cup$		$\cup$
$\nabla S = 0$	$\supset$	$\nabla R = 0$	$\subset$	$\nabla C = 0$
$\cup$		$\cup$		$\cup$
$S = \frac{\kappa}{n} g$	$\supset$	$R = \frac{\kappa}{(n-1)n} G$	$\subset$	$C = 0$

Quasi-Einstein manifolds

A manifold (M, g), $n \geq 3$, is said to be *Einstein manifold* if its Ricci tensor S is proportional to the metric tensor g, i.e.

$$S \; = \; \frac{\kappa}{n}\, g,$$

holds on M, where κ is the scalar curvature of (M, g).

The manifold (M, g), $n \geq 3$, is called the *quasi-Einstein manifold* if at every point $x \in M$ its Ricci tensor S satisfies

$$S \; = \alpha\, g + \varepsilon\, \omega \otimes \omega, \qquad \alpha \in \mathbb{R}, \quad \varepsilon \; = \; \pm 1, \quad \omega \in T^*_x M.$$

- The covector ω is non-zero at every point of the set $\mathcal{U}_S \subset M$.
- If at $x \in \mathcal{U}_S \subset M$ the tensor S has another decomposition of the form

$$S \; = \; \widetilde{\alpha}\, g + \widetilde{\varepsilon}\, \widetilde{\omega} \otimes \widetilde{\omega}, \qquad \widetilde{\alpha} \in \mathbb{R}, \quad \widetilde{\varepsilon} \; = \; \pm 1, \quad \widetilde{\omega} \in T^*_x M$$

then $\alpha \; = \; \widetilde{\alpha}$, $\varepsilon \; = \; \widetilde{\varepsilon}$ and $\omega \; = \; \pm\, \widetilde{\omega}$ at this point.

Quasi-Einstein manifolds - some properties

On a manifold (M, g), $n \geq 3$, we have:

- on $\mathcal{U}_S$ any of the following three conditions is equivalent to each other:

$$\begin{aligned} \mathrm{rank}(S - \alpha\, g) &= 1, \\ (S - \alpha\, g) \wedge (S - \alpha\, g) &= 0, \\ \frac{1}{2}\,(S \wedge S) - \alpha\,(g \wedge S) + \frac{\alpha^2}{2}\,(g \wedge g) &= 0, \end{aligned}$$

where α is some function on $\mathcal{U}_S$,

- if $\mathrm{rank}(S - \alpha\, g) = 1$ holds on $\mathcal{U}_S$ then on this set we have

$$S^2 - \frac{tr(S^2)}{n}\, g = (\kappa - (n-2)\alpha)\,(S - \frac{\kappa}{n}\, g),$$

where α is some function on $\mathcal{U}_S$ and the $(0, 2)$-tensor S^2 is defined by $S^2(X, Y) = S(\mathcal{S}(X), Y)$ and $\mathcal{S}$ is the Ricci operator of (M, g), $X, Y \in \Xi(M)$.

Basic definitions (1)

Let $N_s^{n+1}(c)$, $n \geq 3$, be a semi-Riemannian space of constant curvature $c = \frac{\widetilde{\kappa}}{n(n+1)}$ with signature $(s, n+1-s)$, where $\widetilde{\kappa}$ is its scalar curvature.

Let M be a hypersurface isometrically immersed in $N_s^{n+1}(c)$ and let g be the metric tensor induced on M from the metric of the ambient space and R and κ the Riemann-Christoffel curvature tensor and the scalar curvature, respectively.

Let H and $\mathcal{A}$ be the second fundamental tensor and the shape operator of M, respectively. We have $H(X,Y) = g(\mathcal{A}X, Y)$, for any vectors fields X, Y tangent to M. The $(0,2)$-tensors H^2 and H^3 are defined by

$$H^2(X,Y) = H(\mathcal{A}X, Y),$$
$$H^3(X,Y) = H^2(\mathcal{A}X, Y),$$

respectively.

Basic definitions (2)

Let $\mathcal{U}_H$ be the set of all points of M at which the tensor H^2, defined by $H^2(X, Y) \ = \ H(\mathcal{A}(X), Y)$,
is **not** a linear combination of the second fundamental tensor H and the metric g of M, whereby $\mathcal{A}$ is the shape operator of M and X and Y are vectors tangent to M.

We note that

$$\mathcal{U}_H \subset \mathcal{U}_C \cap \mathcal{U}_S \subset M$$

holds on every hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$.

Basic definitions (3)

The Gauss equation of M in $N_s^{n+1}(c)$ reads

$$R = \frac{\varepsilon}{2} H \wedge H + \frac{\widetilde{\kappa}}{n(n+1)} G, \qquad \varepsilon = \pm 1,$$

where $G = \frac{1}{2} g \wedge g$. From the Gauss equation we get

$$S = \varepsilon \left(tr(H)\, H - H^2 \right) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)} g,$$

and

$$\frac{\kappa}{n-1} - \frac{\widetilde{\kappa}}{n+1} = \frac{\varepsilon}{n-1} \left((tr(H))^2 - tr(H^2) \right).$$

Quasi-umbilical hypersurfaces

Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 3$.
If M is quasi-umbilical at $x \in M$, i.e.

$$H \;=\; \alpha\, g + \varepsilon_1\, \omega \otimes \omega, \qquad \omega \in T_x^* M, \quad \alpha \in \mathbb{R}, \quad \varepsilon_1 = \pm 1,$$

then at this point we have

$$S \;=\; \left(\varepsilon\alpha(tr(H) - \alpha) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)} \right) g + (n-2)\varepsilon\varepsilon_1\alpha\ \omega \otimes \omega.$$

Thus we have the following statement.

• *Every quasi-umbilical hypersurface M in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 3$, is a quasi-Einstein manifold.*

Conformally flat hypersurfaces (1)

According to a well-known Theorem of E. Cartan and J.A. Schouten, a hypersurface M in a conformally flat Riemannian manifold $\widetilde{N}$, $\dim \widetilde{N} \geq 5$, is conformally flat if and only if it is quasi-umbilical ([C],[S]).

This result remains valid when M is a conformally flat hypersurface in a conformally flat semi-Riemannian manifold $\widetilde{N}$, $\dim \widetilde{N} \geq 5$, ([DV]).

[C] E. Cartan, La déformation des hypersurfaces dans l'espace conforme réel a $n \geq 5$ dimensions, Bull. Soc. Math. France 45 (1917), 57–121.

[S] J.A. Schouten, Über die konforme Abbildung n-dimensionaler Mannigfaltigkeiten mit quadratischer Massbestimmung auf eine Mannigfaltigkeit mit euklidischer Massbestimmung, Math. Z. 11 (1921), 58–88.

[DV] R. Deszcz and L. Verstraelen, Hypersurfaces of semi-Riemannian conformally flat manifolds, in: Geometry and Topology of Submanifolds, III, World Sci., River Edge, NJ, 1991, 131–147.

Conformally flat hypersurfaces (2)

From the above presented results we obtain immediately:

Theorem
Every conformally flat hypersurface M isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, is a quasi-Einstein manifold.

Conformally flat hypersurfaces (3)

Lemma ([DG],[DDV]). Let (M, g) be a 3-dimensional semi-Riemannian manifold or a conformally flat semi-Riemannian manifold of dimension ≥ 4. Then on $\mathcal{U}_S \subset M$ the following three conditions are equivalent to each other:

$$\begin{aligned} R \cdot R &= \rho\, Q(g, R), \\ R \cdot S &= \rho\, Q(g, S), \\ S^2 - \frac{tr(S^2)}{n}\, g &= \left(\frac{\kappa}{n-1} + (n-2)\rho\right)\left(S - \frac{\kappa}{n}\, g\right), \end{aligned}$$

where ρ is some function on $\mathcal{U}_S$.

[DG] R. Deszcz and W. Grycak, On certain curvature conditions on Riemannian manifolds, Colloquium Math. 58 (1990), 259–268.

[DDV] J. Deprez, R. Deszcz and L. Verstraelen, Examples of pseudosymmetric conformally flat warped products, Chinese J. Math. 17 (1989), 51–65.

Conformally flat hypersurfaces (4)

Thus we have:

Theorem ([DHV]).
Every conformally flat hypersurface M isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, is a quasi-Einstein pseudosymmetric manifold.
Precisely, if $\mathrm{rank}(S - \alpha\, g) = 1$ on $\mathcal{U}_S \subset M$ then

$$R \cdot R \;=\; (\frac{\kappa}{n-1} - \alpha)\, Q(g, R),$$

holds on $\mathcal{U}_S$, where α is some function on this set.

[DHV] R. Deszcz, S. Haesen and L. Verstraelen, On natural symmetries, in: Topics in Differential Geometry, Romanian Academy of Sciences, Bucharest, 2008, 249-308.

Conformally flat hypersurfaces (5)

From the last theorem we obtain immediately:

Corollary
Let M be hypersurface isometrically immersed in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 3$.
Let $\rho_1,\ \rho_2,\ \ldots,\ \rho_n$ be the eigenvalues of the Ricci operator $\mathcal{S}$ of M.
If at every point of $\mathcal{U}_S \subset M$ we have $\rho_1 = \ldots = \rho_{n-1} \neq \rho_n$ then

$$\operatorname{rank}(S - \rho_1\, g) = 1 \qquad \text{and} \qquad R \cdot R = \frac{\rho_n}{n-1}\, Q(g, R)$$

hold on $\mathcal{U}_S$.

Riemannian 3-manifolds

We mention that 3-dimensional Riemannian manifolds with the Ricci eigenvalues $\rho_1 = \rho_2 \neq \rho_3$ were investigated among others in:

[K] O. Kowalski, A classification of Riemannian 3-manifolds with constant principal Ricci curvatures $\rho_1 = \rho_2 \neq \rho_3$, Nagoya Math. J., 132 (1993), 1–36.

[KN] O. Kowalski and S. Nikcevic, On Ricci eigenvalues of locally homogeneous Riemannian 3-manifolds Geom. Dedicata, 62 (1996), 67–72.

[KS] O. Kowalski and M. Sekizawa, Local isometry classes of Riemannian 3-manifolds with constant Ricci eigenvalues $\rho_1 = \rho_2 \neq \rho_3 > 0$, Archivum Math. (Brno), 32 (1996), 137–145.

[KS] O. Kowalski and M. Sekizawa, Pseudo-symmetric spaces of constant type in dimension three, Personal Note, Charles University - Tokyo Gakugei University, Prague - Tokyo 1998, 1–56.

[HS] N. Hashimoto and M. Sekizawa, Three-dimensional conformally flat pseudo-symmetric spaces of constant type, Arch. Math. (Brno), 36 (2000), 279–286.

3-dimensional semi-Riemannan manifolds

Theorem ([DVY]). Every quasi-Einstein 3-dimensional semi-Riemannian manifold is pseudosymmetric and conversely.

[DVY] R. Deszcz, L. Verstraelen and S. Yaprak, Warped products realizing a certain condition of pseudosymmetry type imposed on the Weyl curvature tensor, Chinese J. Math., 22 (1994), 139–157.

Warped products with 1-dimensional base manifold

Theorem ([Ch-DDGP]).
Every warped product $\overline{M} \times_F \widetilde{N}$ with 1-dimensional base manifold $(\overline{M}, \overline{g})$ and $(n-1)$-dimensional, $n \geq 4$, Einsteinian fibre manifold $(\widetilde{N}, \widetilde{g})$, satisfies on $\mathcal{U}_S$

$$\begin{aligned} \operatorname{rank}\,(S - (\frac{\kappa}{n-1} - L_S)\, g) &= 1, \\ R \cdot S &= L_S\, Q(g, S), \end{aligned}$$

where L_S is some function on $\mathcal{U}_S$.
Moreover, if $n \geq 5$ then on $\mathcal{U}_S \cap \mathcal{U}_C$ we have

$$(n-2)\,(R \cdot C - C \cdot R) = Q(S, R) - L_S\, Q(g, R).$$

[Ch-DDGP] J. Chojnacka-Dulas, R. Deszcz, M. Glogowska, M. Prvanovic, On warped product manifolds satisfying some curvature conditions, to appear.

4-dimensional warped products with 1-dimensional base

Let $(\overline{M}, \overline{g})$ be 1-dimensional manifold, $\overline{g}_{11} = \pm 1$,
and $(\widetilde{N}, \widetilde{g})$ a 3-dimensional semi-Riemannian space of constant curvature.
It is well-known that $\overline{M} \times_F \widetilde{N}$ is a conformally flat manifold.
If $\overline{g}_{11} = -1$ and $(\widetilde{N}, \widetilde{g})$ a 3-dimensional Riemannian space of constant curvature then the warped product $\overline{M} \times_F \widetilde{N}$ is called the *Friedmann-Lemaitre-Robertson-Walker spacetime*.
Thus we have

Theorem

The Friedmann-Lemaitre-Robertson-Walker spacetimes are quasi-Einstein, pseudosymmetric and conformally flat manifolds.

Conclusion

We have the following conclusion

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and investigation in quasi-umbilical hypersurfaces of conformally flat spaces.

On some class of generalized Robertson-Walker spacetimes (1)

Let $\overline{M} \times_F \widetilde{N}$ be the warped product
with 1-dimensional manifold $(\overline{M}, \overline{g})$, $\overline{g}_{11} = \varepsilon = \pm 1$,
and $(n-1)$-dimensional semi-Riemannian manifold $(\widetilde{N}, \widetilde{g})$, $n - 1 \geq 3$,
with the warping function F defined by

$$\begin{aligned} F(x^1) &= \varepsilon C_1 (x^1 + \frac{\varepsilon c}{C_1})^2 , \quad \varepsilon C_1 > 0 , \\ F(x^1) &= \frac{c}{2}(\exp(\pm\frac{b}{2}x^1) - \frac{2\varepsilon C_1}{b^2 c} \exp(\mp\frac{b}{2}x^1))^2 , \quad c > 0 , \quad b \neq 0 , \\ F(x^1) &= \frac{2\varepsilon C_1}{c^2}(1 + \sin(cx^1 + b)) , \quad \varepsilon C_1 > 0 , \quad c \neq 0 , \end{aligned} \tag{1}$$

where b, c and C_1 are constants and $x^1 \in \overline{M}$.
The functions F, defined by (1), satisfy the following differential equation

$$F\, F^{''} - (F^{'})^2 + 2\varepsilon C_1\, F = 0, \qquad F^{'} = \frac{dF}{dx^1}, \quad F^{''} = \frac{dF^{'}}{dx^1}.$$

On some class of generalized Robertson-Walker spacetimes (2)

Further, let $(\widetilde{N}, \widetilde{g})$ be a quasi-Einstein manifold. Precisely, let

$$\operatorname{rank}(\widetilde{S} - (n-2)C_1 \widetilde{g}) = 1$$

be satisfied on $\mathcal{U}_{\widetilde{S}}$. Now we can check that

$$\begin{aligned} \operatorname{rank}(S - \frac{\kappa}{n} g) &= 1, \\ R \cdot S &= \frac{\kappa}{(n-1)n} Q(g, S) \end{aligned}$$

hold on $\mathcal{U}_S \subset \overline{M} \times_F \widetilde{N}$ ([Ch-DDGP]). We refer to [DH](Example 5.1) for an example of such quasi-Einstein Ricci-pseudosymmetric manifold.

[Ch-DDGP] J. Chojnacka-Dulas, R. Deszcz, M. Glogowska, M. Prvanovic, On warped product manifolds satisfying some curvature conditions, to appear.

[DH] R. Deszcz and M. Hotlos, On some pseudosymmetry type curvature condition, Tsukuba J. Math. 27 (2003), 13–30.

Quasi-Einstein hypersurfaces (1)

Proposition ([DHS]).
Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying

$$S = \alpha\, g + \varepsilon\, \omega \otimes \omega\,, \quad \varepsilon = \pm 1$$

on $\mathcal{U}_H \subset M$ for some function α and 1-form ω on $\mathcal{U}_H$.
(i) Then on $\mathcal{U}_H$ we have

$$\mathcal{A}(W) = \beta\, W,$$
$$H^3 = (tr(H) - \beta)\, H^2 + \gamma\, H + \frac{(n-1)\varepsilon\beta\widetilde{\kappa}}{n(n+1)}\, g,$$

where $\mathcal{A}$ is the shape operator of M and W is the vector field on $\mathcal{U}_H$ such that $\omega(X) = g(W, X)$ and β and γ are some functions on $\mathcal{U}_H$.

[DHS] R. Deszcz, M. Hotlos and Z. Senturk, On curvature properties of certain quasi-Einstein hypersurfaces, International J. Math., 23 (2012), 1250073, 17 pages.

Quasi-Einstein hypersurfaces (2)

(ii) Moreover, if on $\mathcal{U}_H$ the following identity is satisfied

$$\sum_{(X_1,X_2),\,(X_3,X_4),\,(X_5,X_6)} (R \cdot C)(X_1, X_2, X_3, X_4; X_5, X_6) \;=\; 0,$$

where $X_1, \ldots, X_6 \in \Xi(M)$, then on $\mathcal{U}_H$ we have: $\mathcal{A}(W) \;=\; 0,$

$$\begin{aligned}
\alpha &= \frac{\kappa}{n-1} - \frac{\widetilde{\kappa}}{n(n+1)}, \\
\|\omega\|^2 &= \varepsilon\Big(\frac{\widetilde{\kappa}}{n+1} - \frac{\kappa}{n-1}\Big), \\
R \cdot S &= \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, S), \\
(n-2)\,(R \cdot C - C \cdot R) &= Q(S, R) - \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, R).
\end{aligned}$$

Quasi-Einstein hypersurfaces (3)

Theorem ([Ch-DDGP]).
On any Ricci-pseudosymmetric manifold (M, g), $n \geq 4$, the condition

$$\sum_{(X_1,X_2),\,(X_3,X_4),\,(X_5,X_6)} (R \cdot C)(X_1, X_2, X_3, X_4; X_5, X_6) \;=\; 0$$

is satisfied, where $X_1, \ldots, X_6 \in \Xi(M)$.

[Ch-DDGP] J. Chojnacka-Dulas, R. Deszcz, M. Glogowska, M. Prvanovic, On warped product manifolds satisfying some curvature conditions, to appear.

Quasi-Einstein hypersurfaces (4)

Proposition. Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying

$$S \ = \ \alpha\, g + \varepsilon\, \omega \otimes \omega\, , \quad \varepsilon \ = \ \pm 1$$

on $\mathcal{U}_H \subset M$ for some function α and 1-form ω on $\mathcal{U}_H$. In addition, if $R \cdot S \ = \ L_S\, Q(g, S)$ holds on $\mathcal{U}_H$ then the following relations are satisfied:

$$\sum_{(X_1,X_2),\,(X_3,X_4),\,(X_5,X_6)} (R \cdot C)(X_1, X_2, X_3, X_4; X_5, X_6) \ = \ 0,$$

$$\mathcal{A}(W) \ = \ 0, \qquad \alpha \ = \ \frac{\kappa}{n-1} - \frac{\widetilde{\kappa}}{n(n+1)},$$

$$\|\omega\|^2 \ = \ \varepsilon\left(\frac{\widetilde{\kappa}}{n+1} - \frac{\kappa}{n-1}\right),$$

$$R \cdot S \ = \ \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, S),$$

$$(n-2)\,(R \cdot C - C \cdot R) \ = \ Q(S, R) - \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, R).$$

Quasi-Einstein hypersurfaces (5)

Remarks. Let M be a hypersurface in a semi-Riemannian space of constant curvature.
(i) An example of a non-pseudosymmetric Ricci-pseudosymmetric quasi-Einstein hypersurface M of dimension ≥ 5 is given in [DHS].
(ii) On 4-dimensional hypersurfaces M the conditions of pseudosymmetry and Ricci-pseudosymmetry are equivalent ([DDSVY]).
(iii) Quasi-Einstein Ricci-pseudosymmetric hypersurfaces M, $\dim M \geq 4$, satisfying some curvature conditions of pseudosymmetry type were investigated in [G].

[DHS] R. Deszcz, M. Hotlos and Z. Senturk, On curvature properties of certain quasi-Einstein hypersurfaces, International J. Math., 23 (2012), 1250073, 17 pages.
[DDSVY] F. Defever, R. Deszcz, Z. Senturk, L. Verstraelen and S. Yaprak, On a problem of P. J. Ryan, Glasgow Math. J. 41 (1999), 271–281.
[G] M. Glogowska, On quasi-Einstein Cartan type hypersurfaces, J. Geom. Phys. 58 (2008) 599–614.