THEORY OF INFINITELY NEAR POINTS IN SMOOTH MANIFOLDS: THE FERMAT FUNCTOR

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The work [3] of A. Weil on infinitesimal prolongations of smooth manifolds had a great influence on Differential Geometry, inspiring research threads like Weil functors, Synthetic Differential Geometry, Differential Geometry over general base fields and rings, but also Grothendieck's approach to infinitesimal neighborhoods in algebraic geometry.

We present a new approach to the extension of smooth manifolds with infinitely near points which permits to formalize infinitesimal methods in Differential Geometry and has a clear geometrical meaning. In case of a smooth manifold M this extension can be easily formulated. We firstly have to introduce *little-oh polynomials* as maps $x : \mathbb{R}_{\geq 0} \longrightarrow M$ that can be written as $\varphi(x(t)) = r + \sum_{i=1}^{k} \alpha_i \cdot t^{a_i} + o(t)$, where $r, \alpha_i \in \mathbb{R}^n$, $a_i \in \mathbb{R}_{\geq 0}$ and $t \to 0^+$, in some chart (U, φ) such that $x(0) \in U$. We can hence introduce an equivalence relation between little-oh polynomials saying that $x \sim y$ iff we can write $\varphi(x(t)) = \varphi(y(t)) + o(t)$ in some chart (U, φ) such that $x(0), y(0) \in U$. The extension of M with infinitely near points is simply the quotient set ${}^\bullet M := M/\sim$. This construction applied to $M = \mathbb{R}$ gives the so-called *ring of Fermat reals* ${}^\bullet\mathbb{R}$, a non-archimedean ring with nilpotent infinitesimals, see e.g. [1, 2].

However, this construction can be generalized to any diffeological space $X \in$ **Diff**, obtaining the *Fermat functor* ${}^\bullet(-) : \mathbf{Diff} \longrightarrow {}^\bullet\boldsymbol{\mathcal{C}}^\infty$ from the category **Diff** of diffeological spaces to that of *Fermat spaces* ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$. Since the category **Diff** is cartesian closed, complete and co-complete, and embeds the category **Man** of smooth manifolds, the whole construction can be applied also to infinite dimensional spaces like the space of all the smooth maps between two smooth manifolds and to spaces with singularities. In this setting we can define a tangent vector as an infinitesimal smooth curve $t : D \longrightarrow X$, where $D = \{h \in {}^\bullet\mathbb{R} \mid h^2 = 0\}$ is the ideal of first order nilpotent infinitesimals; we can see any vector field as an infinitesimal transformation of the space X into itself and we can prove that any vector field has a unique infinitesimal integral curve. For the case ${}^\bullet\boldsymbol{\mathcal{C}}^\infty({}^\bullet M, {}^\bullet N)$ this amount to say that an infinite system of ODE has always a unique infinitesimal solution.

Even if the whole construction does not need any background of Mathematical Logic, the study of the preservation properties of the Fermat functor ${}^\bullet(-)$ reveals a surprising strong connection with intuitionistic logic. In fact, this functor preserves product of manifolds, open subspaces, inclusion, inverse images, intersections and unions, intuitionistic negation and implication, intuitionistic quantifiers. Therefore, a full transfer theorem for intuitionistic formulas holds and summarizes the preservation properties of this functor.

[1] P. Giordano, *The ring of fermat reals*, Advances in Mathematics 225, pp. 2050–2075, 2010.

[2] P. Giordano, *Fermat-Reyes method in the ring of Fermat reals.* Advances in Mathematics 228, pp. 862–893, 2011.

[3] A. Weil, *Théorie des points proches sur les variétés différentiables.* Colloque de Géometrie Différentielle, Strasbourg, pp. 111–117, 1953.