TALASIĆI (WAVELETS)

1. Transformacija
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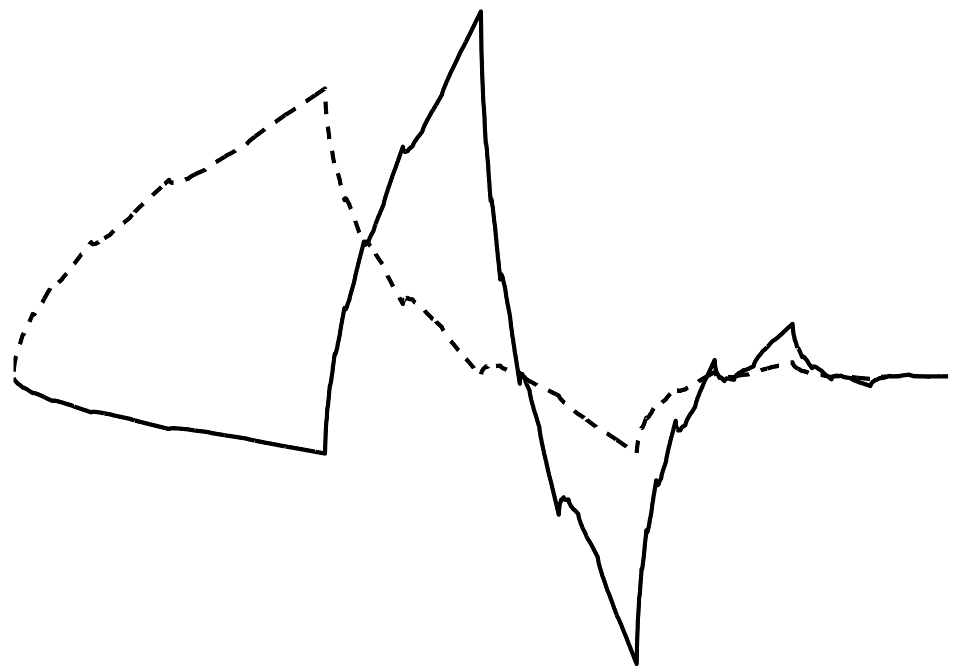
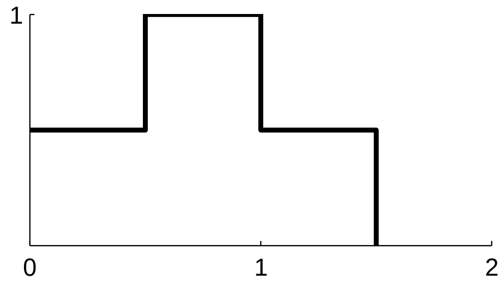
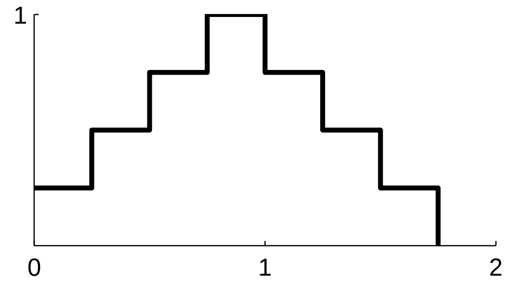


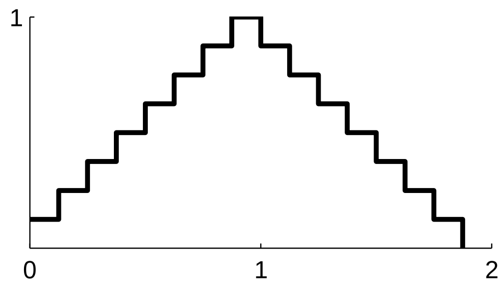
2007

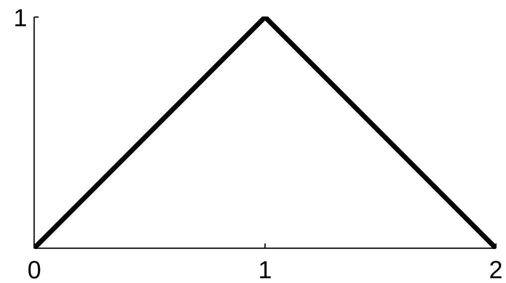
Kaskadni algoritam $$\lim_{n\to\infty} \varphi^{(n)}(x) = \varphi(x)$$

$$\varphi^{(0)}(x) = \aleph_{[0,1)}(x), \qquad \varphi^{(n+1)}(x) = \sum_{k=0}^{N-1} c(k)\sqrt{2}\varphi^{(n)}(2x-k), \quad n = 0, 1, \ldots$$









Krov funkcija

$$\varphi^{(n+1)}(x) = \frac{1}{2}\varphi^{(n)}(2x) + \varphi^{(n)}(2x-1) + \frac{1}{2}\varphi^{(n)}(2x-2)$$

Algoritam zasnovan na Fourier-ovoj transformaciji

$$\widehat{\varphi}(\omega) = \int \varphi(x)\, e^{\imath\omega x}\, dx = \sum_{k=0}^{N-1} c(k)\sqrt{2} \int \varphi(2x-k)\, e^{\imath\omega x}\, dx$$

$$= \left(\frac{1}{\sqrt{2}} \sum_{k=0}^{N-1} c(k) e^{\imath \frac{k\omega}{2}} \right) \int \varphi(y)\, e^{\imath \frac{y\omega}{2}}\, dy = H\big(\frac{\omega}{2}\big)\, \widehat{\varphi}\big(\frac{\omega}{2}\big)$$

$$H(\omega) = \frac{1}{\sqrt{2}} \sum_{k=0}^{N-1} c(k) e^{\imath k\omega}, \qquad H(0) = 1, \qquad \widehat{\varphi}(0) = \int \varphi(x)\, dx = 1$$

$$\widehat{\varphi}(\omega) = \left(\prod_{j=1}^{n} H\big(\frac{\omega}{2^j}\big) \right) \widehat{\varphi}\big(\frac{\omega}{2^n}\big) \quad \xrightarrow{n\to\infty} \quad \prod_{j=1}^{\infty} H\big(\frac{\omega}{2^j}\big)$$

Algoritam zasnovan na rekurziji

$$\varphi(j) = \sum_{k=0}^{N-1} c(k)\sqrt{2}\varphi(2j-k), \qquad j = 0,\dots,N-2,$$

$$\varphi\left(\frac{2j+1}{2}\right) = \sum_{k=0}^{N-1} c(k)\sqrt{2}\varphi\big((2j+1)-k\big)$$

$$\varphi\left(\frac{2j+1}{4}\right) = \sum_{k=0}^{N-1} c(k)\sqrt{2}\varphi\left(\frac{2j+1}{2}-k\right)$$

$$\vdots$$

Db2

$$\frac{1}{4}\begin{pmatrix} 1+\sqrt{3} & 0 & 0 \\ 3-\sqrt{3} & 3+\sqrt{3} & 1+\sqrt{3} \\ 0 & 1-\sqrt{3} & 3-\sqrt{3} \end{pmatrix} \begin{pmatrix} \varphi(0) \\ \varphi(1) \\ \varphi(2) \end{pmatrix} = \begin{pmatrix} \varphi(0) \\ \varphi(1) \\ \varphi(2) \end{pmatrix} \longrightarrow \begin{matrix} \varphi(0) = 0 \\ \varphi(1) = (1+\sqrt{3})/2 \\ \varphi(2) = (1-\sqrt{3})/2 \end{matrix}$$

♠ *Funkcija skaliranja $\varphi(x)$, određena sa N koeficijenata, ima kompaktan nosač na intervalu* $[0,\ N-1]$.

$$\varphi(x) \neq 0 \quad x \in [a,b] \quad \longrightarrow \quad \varphi(2x-k) \neq 0 \quad x \in [(a+k)/2,\ (b+k)/2]$$

Nosači funkcija na levoj i desnoj strani dilatacione jednačine su identični

$$[a,\ b] = \cup_{k=0}^{N-1} \left[\frac{a+k}{2}, \frac{b+k}{2}\right] = \left[\frac{a}{2}, \frac{b+N-1}{2}\right] \quad \longrightarrow \quad a=0,\ b=N-1=N1$$

$$\varphi^{(0)}(x) \quad [0,1], \qquad \varphi^{(1)}(x) \quad [0,\ (1+N1)/2], \qquad \varphi^{(2)}(x) \quad [0,\ (1+3N1)/4]$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\varphi^{(n)}(x) \quad [0,\ \big(1+(2^n-1)N1\big)/2^n] \quad \stackrel{n\to\infty}{\longrightarrow} \quad \varphi(x) \quad x \in [0,N1] \equiv [0,N-1]$$

Dužina nosača talasića određena je brojem nenula koeficijenata $d(n)$.

♠ *Pretpostavimo da kaskadni algoritam konvergira,* $\varphi^{(i)}(x) \to \varphi(x)$, *uniformno po* x. *Ako koeficijenti* $c(n)$ *i* $d(n)$ *zadovoljavaju uslove*

$$\begin{aligned} \sum c(n)c(n-2m) &= \delta(m), \\ \sum d(n)d(n-2m) &= \delta(m), \\ \sum c(n)d(n-2m) &= 0, \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} & d(n) = (-1)^n c(N-1-n), \\ & n = 0, \ldots, N-1, \quad N \text{ parno} \end{aligned}$$

slede ortogonalnosti

(i) $$\int_{-\infty}^{\infty} \varphi(x-n)\varphi(x-m)\,dx = \delta(n-m)$$

(ii) $$\int_{-\infty}^{\infty} \varphi(x-n)\psi(x-m)\,dx = 0$$

(iii) $$\int_{-\infty}^{\infty} \psi_{jk}(x)\psi_{JK}(x)\,dx = \delta(j-J)\delta(k-K)$$

(i) Dokaz izvodimo indukcijom. Četvrtka je ortogonalna u odnosu na translaciju (nosači se ne preklapaju), pa $\varphi^{(0)}(x-n)$ čine ortonormirani sistem funkcija. Dalje, za $l_1 = l - 2(m-n)$,

$$\int_{-\infty}^{\infty} \varphi^{(i+1)}(x-m)\varphi^{(i+1)}(x-n)\,dx$$

$$= \int \left(\sqrt{2}\sum_k c(k)\varphi^{(i)}\big(2(x-m)-k\big)\right)\left(\sqrt{2}\sum_l c(l)\varphi^{(i)}\big(2(x-n)-l\big)\right)dx$$

$$= 2\int \left(\sum_k c(k)\varphi^{(i)}(2x-2m-k)\right)\left(\sum_{l_1} c(l_1-2(n-m))\varphi^{(i)}(2x-2m-l_1)\right)dx$$

$$= \sum_k \sum_{l_1} c(k)c(l_1-2(n-m)) \int \varphi^{(i)}(2(x-m)-k)\,\varphi^{(i)}(2(x-m)-l_1)\,d(2x)$$

$$= \sum_k c(k)c(k-2(n-m)) = \delta(n-m)$$

(ii) Na osnovu dokazane ortogonalnosti $\varphi(x-n)$ je

$$\int_{-\infty}^{\infty} \varphi(x-m)\psi(x-n)\,dx$$

$$= \int \Big(\sqrt{2}\sum_k c(k)\varphi\big(2(x-m)-k\big)\Big)\Big(\sqrt{2}\sum_l d(l)\varphi\big(2(x-n)-l\big)\Big)\,dx$$

$$= 2\int \Big(\sum_k c(k)\varphi(2x-2m-k)\Big)\Big(\sum_{l_1} d(l_1-2(n-m))\varphi(2x-2m-l_1)\Big)\,dx$$

$$= \sum_k \sum_{l_1} c(k)d(l_1-2(n-m)) \int \varphi(2(x-m)-k)\,\varphi(2(x-m)-l_1)\,d(2x)$$

$$= \sum_k c(k)d(k-2(n-m)) = 0$$

(iii) Ortogonalnost talasića na istom nivou rezolucije (za isto j) sledi iz ortogonalnosti $\varphi(x-n)$

$$\begin{aligned}
&\int_{-\infty}^{\infty} \psi(x-m)\psi(x-n)\,dx \\
&= \int \Big(\sqrt{2}\sum_k d(k)\varphi\big(2(x-m)-k\big)\Big)\Big(\sqrt{2}\sum_l d(l)\varphi\big(2(x-n)-l\big)\Big)\,dx \\
&= \sum_k d(k)d(k-2(n-m)) = \delta(n-m)
\end{aligned}$$

Ortogonalnost talasića za različite nivoe rezolucije sledi iz multirezolucije.
Za $j \neq J$ je $\mathcal{W}_J \perp \mathcal{W}_j$, jer je

$$\mathcal{V}_J \oplus \mathcal{W}_J = \mathcal{V}_{J-1} \subset \mathcal{V}_{J-2} \subset \cdots \subset \mathcal{V}_j, \quad \longrightarrow \quad \mathcal{W}_J \subset \mathcal{V}_j \perp \mathcal{W}_j$$

Funkcije skaliranja na različitim rezolucijskim nivoima nisu ortogonalne.

Talasić je funkcija talasnog oblika (oscilatorna) koja ima ograničeno trajanje. Njena srednja vrednost je nula.

Možemo ga definisati zadavanjem

1. *prostora* $\mathcal{W}_j$, kao razlike prostora multirezolucije $\mathcal{V}_j$
2. *talasića* $\psi(x)$, tako što direktno biramo funkciju navedenih osobina
3. *koeficijenata* $c(n)$ i $d(n) = (-1)^n c(N-1-n)$, pa nalazimo funkciju skaliranja koja određuje talasić $\psi(x)$

Prikažimo sva tri pristupa na primeru četvrtke i ortogonalnog talasića

$$\varphi(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & x \notin [0,1) \end{cases} \qquad \psi(x) = \begin{cases} 1, & x \in [0,1/2) \\ -1, & x \in [1/2,1) \\ 0, & x \notin [0,1) \end{cases}$$

1. *prostor*

$$\mathcal{V}_0 = \{f(x) \,|\, f(x) = f(n),\, x \in [n, n+1)\},$$
$$\mathcal{V}_{-1} = \{f(x) \,|\, f(x) = f(n/2),\, x \in [n/2, (n+1)/2)\}$$

$\mathcal{W}_0 \subset \mathcal{V}_{-1}$, te sadrži funkcije jednake konstantama na polovinama intervala.

$$\mathcal{W}_0 \perp \mathcal{V}_0 \qquad \forall f \in \mathcal{W}_0, \quad \forall g \in \mathcal{V}_0, \qquad \longrightarrow$$

$$\begin{aligned} 0 = (f, g) &= \sum_n \int_n^{n+1} f(x) g(x)\, dx \\ &= \sum_n g(n) \int_n^{n+1} f(x)\, dx \\ &= \frac{1}{2} \sum_n g(n)\big(f(n) + f(n+1/2)\big) \end{aligned}$$

Sledi da je

$$\mathcal{W}_0 = \{f(x) \,|\, f(x) = \begin{cases} f(n), & x \in [n, n+1/2), \\ -f(n), & x \in [n+1/2, n+1) \end{cases}\}$$

2. *talasić* – A. Haar (1909) $\psi(x) = \begin{cases} 1, & x \in [0, 1/2) \\ -1, & x \in [1/2, 1) \\ 0, & x \notin [0, 1) \end{cases}$

$$(\psi_{j,m}, \varphi_{j,n}) = 0, \qquad (\psi_{j,m}, \psi_{J,n}) = \delta(j-J)\,\delta(m-n)$$

Translacije $\psi(x-k)$ generišu prostor $\mathcal{W}_0$, a $\psi(2^{-j}x-k)$ prostor $\mathcal{W}_j$.

$\{\psi_{j,k}(x)\}_{j,k}$, $j, k \in \mathcal{Z}$ je ortogonalni bazis u $\mathcal{L}_2$,

$$f(x) = \sum_{j,k} (f, \psi_{jk}) \psi_{jk}(x), \qquad f \in \mathcal{L}_2$$

3. *koeficijenti* $c(0) = c(1) = 1/\sqrt{2}$ definišu četvrtku.
Obrnuti poredak sa alternativnom promenom znaka definiše koeficijente talasića

$$d(0) = 1/\sqrt{2}, \qquad d(1) = -1/\sqrt{2}$$