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MODIFIED EQUIPARTITION CALCULATION FOR SUPERNOVA REMNANTS

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ABSTRACT

Determination of the magnetic field strength in the interstellar medium is one of the more complex tasks of contemporary astrophysics. We can only estimate the order of magnitude of the magnetic field strength by using a few very limited methods. Besides the Zeeman effect and Faraday rotation, the equipartition or minimum-energy calculation is a widespread method for estimating magnetic field strength and energy contained in the magnetic field and cosmic-ray particles by using only the radio synchrotron emission. Despite its approximate character, it remains a useful tool, especially when there are no other data about the magnetic field in a source. In this paper, we give a modified calculation that we think is more appropriate for estimating magnetic field strengths and energetics in supernova remnants (SNRs). We present calculated estimates of the magnetic field strengths for all Galactic SNRs for which the necessary observational data are available. The Web application for calculation of the magnetic field strengths of SNRs is available at http://poincare.matf.bg.ac.rs/~arbo/eqp/.

Key words: ISM: magnetic fields – ISM: supernova remnants – radio continuum: general

1. INTRODUCTION

The basic constituents of the interstellar medium (ISM) are normal (thermalized) particles, cosmic rays (CRs), radiation, and the magnetic field. Each of these four forms of ISM contains similar energy density of about 1 eV cm^{-3}. If we compare the quantity of information available for each of them, we can immediately conclude that the magnetic field is absolutely the most intriguing and hidden form of ISM. Recent simulations of supernova remnant (SNR) shocks commonly include the magnetic field because it plays an important part in various related phenomena (particle acceleration, radiation, shock compression and formation, etc.). The magnetic field strength and its direction can only be approximately estimated by using a few methods that are very limited in their applicabilities (for recent review of magnetic fields in SNRs see Reynolds et al. 2011). One of these is the Zeeman effect, which is an appropriate method for generally stronger fields and can be used for the determination of strong ISM magnetic fields in high-density H I or molecular clouds rich in OH and CN. The global magnetic field of the Galaxy, a few μG, is too small to be measured in this way. The second method for determination of the component of ISM magnetic field parallel to the line of sight is the so-called Faraday rotation or rotation measure method. The rotation measure (RM) is calculated directly from the radio astronomical polarization observations at multiple frequencies. This quantity depends upon the plasma density and the strength of the field component along the line of sight. Under necessary simplistic assumptions, the RM can yield an order of magnitude estimate of the magnetic field strength between the source and observer. If several distinct rotating regions located along the line of sight generate a spectrum of various RM components, multi-channel spectropolarimetric radio data that can be Fourier-transformed into Faraday space, called RM synthesis (see Heald 2009; Beck 2011 and references therein), are needed. If we would like to estimate the magnetic field strength directly connected to a source embedded in the relatively low-density region, the only way is by using the so-called equipartition calculation.

The equipartition or minimum-energy calculation is a widespread method for estimating magnetic field strength and energy contained in the magnetic field and CR particles using only the radio synchrotron emission of a source. Despite its approximate character, it remains a useful tool in situations when no other data about the source are available. Details of equipartition and revised equipartition calculations for radio sources in general are available in Pacholczyk (1970, hereafter P70), Govoni & Feretti (2004), and Beck & Krause (2005, hereafter BK05), respectively. A discussion on whether equipartition of energy is fulfilled in real sources, and how reliable magnetic field estimates from equipartition calculation are, can be found in Duric (1990).

In his famous book, Pacholczyk gave the fundamental concepts of the equipartition or minimum-energy calculation. The first ingredient of the equipartition calculation is an expression for the total energy of relativistic particles, which can be obtained by an integration of power-law energy distribution of CRs. The total energy of relativistic particles was found by integration over all frequencies in the radio domain. Pacholczyk assumed a homogenous magnetic field for the calculation of energy contained in the magnetic field, and a coefficient $\mathtt{K}$ which represents the ratio between energies of relativistic protons and electrons. The last ingredient in the P70 equipartition formula is the radio luminosity of an object.

BK05 presented a revised equipartition calculation. The basic improvement in comparison to the classical P70 equipartition is the integration of the power-law energy distribution over energies instead of over frequencies. They integrated over two energy ranges with a break at $E = mc^2$, where m is the rest mass of the accelerated particles, i.e., two power-law distributions with different slopes, both dependent on energy spectral index γ. Instead of luminosity as used in the classical approach, BK05 used radio intensity—their intention was to determine the magnetic field strength of a small part of a very extended object such as the whole Galaxy or an extragalactic system. The magnetic field small-scale structures of very extended objects are very far from being homogenous. The model of magnetic field distribution used in the revised equipartition formula accommodates such extended objects. Finally, BK05 used a coefficient $\mathbf{K_0}$, which represents the ratio of the number densities of CR protons to electrons, instead of the ratio between energies of protons and electrons used in the classical equipartition.

In this paper, we use the energy ratio, as in the classical calculation, but it includes all heavier particles that can be found in CRs. Also, we use the radio flux density instead of the radio luminosity used in P70 equipartition or the specific intensity from the revised calculation of BK05. Since our intention is to derive equipartition formulas for the determination of the magnetic fields and the minimal energies in SNRs, we use a model of the magnetic field distribution defined in Longair (1994). Finally, since the distribution of CRs is a power law in momentum (which, for sufficiently high energies, can be transformed to the same power law in energy), we have chosen to integrate over momentum and not over energies as BK05 did, so there is no need to introduce the break in the differential energy spectrum.

We emphasize that the final formulas in the P70 equipartition do not depend on the energy spectral index (or radio spectral index, $\alpha = (\gamma - 1)/2$), while in the BK05 and our equipartition these formulas depend on the energy spectral index (see Equations (12) and (13)).

In the following section, by relying on Bell's theory of diffusive shock acceleration (DSA; Bell 1978a, 1978b) and his assumption concerning injection of particles into the acceleration process, we will first derive a modified equipartition, i.e., a minimum-energy calculation (Arbutina et al. 2011) applicable to "mature" SNRs ($v_s \ll$ 6000–7000 km s^{-1}) with radio spectral index $0.5 < \alpha < 1$ (energy spectral index $2 < \gamma < 3$). Then we will incorporate the dependence $\epsilon = \epsilon(E_{\rm inj})$ which will make the formula applicable to younger, i.e., all, SNRs.

2. ANALYSIS AND RESULTS

2.1. A Simple Approach

Following Bell (1978b) we will assume that a certain number of particles have been injected into the acceleration process, all with the same injection energy $E_{\rm inj} \approx 4\frac{1}{2}m_p v_s^2$.[4] If we assume that the shock velocity is low enough so that $E_{\rm inj} \ll m_e c^2$ (and $p^e_{\rm inj} \ll m_e c$), for the energy density of a CR species (e.g., electrons, protons, α-particles, heavier ions), assuming a power-law momentum distribution, we have

$$\begin{aligned}\epsilon &= \int_{p_{\rm inj}}^{p_\infty} 4\pi k p^{-\gamma}\left(\sqrt{p^2c^2+m^2c^4} - mc^2\right)dp \\ &\approx \int_0^\infty 4\pi k p^{-\gamma}\left(\sqrt{p^2c^2+m^2c^4} - mc^2\right)dp \\ &= 4\pi k c (mc)^{2-\gamma}\int_0^\infty x^{-\gamma}\left(\sqrt{x^2+1}-1\right)dx, \quad x = \frac{p}{mc} \\ &= K(mc^2)^{2-\gamma}\frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}, \quad K = 4\pi k c^{\gamma-1}, \quad 2<\gamma<3,\end{aligned} \tag{1}$$

where k is the constant in the distribution function $f(p) = kp^{-(\gamma+2)}$. The function under the integral in Equation (1) is approximately a power law with spectral index of $2 - \gamma$ for thermal (non-relativistic) particles and a power law with spectral index of $1 - \gamma$ for highly relativistic particles. In this paper, the sharp break in BK05 is replaced by a smooth one.

The total CR energy density is then

$$\begin{aligned}\epsilon_{\rm CR} &= \frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}\left(K_e(m_ec^2)^{2-\gamma} + \sum_i K_i(m_ic^2)^{2-\gamma}\right) \\ &= \frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}\Bigg(K_e(m_ec^2)^{2-\gamma} + K_p(m_pc^2)^{2-\gamma} \\ &\quad\times\sum_i \frac{n_i}{n_p}\left(\frac{m_i}{m_p}\right)^{(3-\gamma)/2}\Bigg) = \frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}K_e(m_ec^2)^{2-\gamma} \\ &\quad\times\left(1 + \frac{n}{n_e}\left(\frac{m_p}{m_e}\right)^{(3-\gamma)/2}\sum_i\frac{n_i}{n}\left(\frac{m_i}{m_p}\right)^{(3-\gamma)/2}\right) \\ &= K_e(m_ec^2)^{2-\gamma}\frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}(1+\kappa),\end{aligned} \tag{2}$$

where

$$\kappa = \left(\frac{m_p}{m_e}\right)^{(3-\gamma)/2}\frac{\sum_i A_i^{(3-\gamma)/2}\nu_i}{\sum_i Z_i\nu_i}, \tag{3}$$

κ represents the energy ratio between ions and electrons, $n_e = \sum_i Z_i n_i$, $\nu_i = n_i/n$ are ion abundances, and A_i and Z_i are mass and charge numbers of elements. We assumed that at high energies $K_p/K_e \approx (n_p/n_e)(m_p/m_e)^{(\gamma-1)/2}$ (see Equation (26)), where K_p and K_e are the constants in the power-law energy distributions for protons and electrons, respectively. Note that we have neglected energy losses.

The emission coefficient for synchrotron radiation is

$$\varepsilon_\nu = c_5 K_e (B\sin\Theta)^{(\gamma+1)/2}\left(\frac{\nu}{2c_1}\right)^{(1-\gamma)/2}, \tag{4}$$

where c_1, c_3, and $c_5 = c_3\Gamma(\frac{3\gamma-1}{12})\Gamma(\frac{3\gamma+19}{12})/(\gamma+1)$ are defined in P70.[5] We will use the flux density defined as

$$S_\nu = \frac{L_\nu}{4\pi d^2} = \frac{\mathcal{E}_\nu V}{4\pi d^2} = \frac{\frac{4\pi}{3}R^3 f\mathcal{E}_\nu}{4\pi d^2} = \frac{4\pi}{3}\varepsilon_\nu f\theta^3 d, \tag{5}$$

where L_ν is the radio luminosity, $\mathcal{E}_\nu$ is the volume emissivity, V is the volume, f is the volume filling factor of radio emission, R is the radius, d is the distance, and $\theta = R/d$ is the angular radius.

If we assume an isotropic distribution for the orientation of pitch angles (Longair 1994), we can take for the average $\langle(\sin\Theta)^{(\gamma+1)/2}\rangle$

$$\frac{1}{2}\int_0^\pi(\sin\Theta)^{(\gamma+3)/2}d\Theta = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{\gamma+5}{4}\right)}{\Gamma\left(\frac{\gamma+7}{4}\right)}. \tag{6}$$

For the total energy, we have

$$E = \frac{4\pi}{3}R^3 f(\epsilon_{\rm CR} + \epsilon_B), \quad \epsilon_B = \frac{1}{8\pi}B^2, \tag{7}$$

$$E = \frac{4\pi}{3}R^3 f\left(K_e(m_ec^2)^{2-\gamma}\frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}(1+\kappa) + \frac{1}{8\pi}B^2\right). \tag{8}$$

[4] We assume a fully ionized, globally electro-neutral plasma.

[5] Namely, $c_1 = 6.264 \times 10^{18}$ and $c_3 = 1.866 \times 10^{-23}$ in cgs units.

Looking for the minimum energy with respect to B, $dE/dB = 0$ gives

$$\frac{dK_e}{dB}(m_e c^2)^{2-\gamma}\frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}(1+\kappa)+\frac{1}{4\pi}B=0, \tag{9}$$

where (by using Equations (4), (5), and (6))

$$\frac{dK_e}{dB}=-\frac{3}{4\pi}\frac{S_\nu}{f\theta^3 d}\frac{1}{c_5}\left(\frac{\nu}{2c_1}\right)^{-(1-\gamma)/2}\frac{(\gamma+1)\Gamma\left(\frac{\gamma+7}{4}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma+5}{4}\right)}B^{-(\gamma+3)/2}, \tag{10}$$

i.e., the magnetic field for the minimum energy is

$$B=\left(\frac{3}{2\pi}\frac{(\gamma+1)\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)\Gamma\left(\frac{\gamma+7}{4}\right)}{(\gamma-1)\Gamma\left(\frac{\gamma+5}{4}\right)}\frac{S_\nu}{fd\theta^3}\right.$$
$$\left.\times\ (m_e c^2)^{2-\gamma}\frac{(2c_1)^{(1-\gamma)/2}}{c_5}(1+\kappa)\nu^{(\gamma-1)/2}\right)^{2/(\gamma+5)} \tag{11}$$

or

$$B\ [\mathrm{G}]\approx\left(6.286\times 10^{(9\gamma-79)/2}\frac{\gamma+1}{\gamma-1}\frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)\Gamma\left(\frac{\gamma+7}{4}\right)}{\Gamma\left(\frac{\gamma+5}{4}\right)}\right.$$
$$\times\ (m_e c^2)^{2-\gamma}\frac{(2c_1)^{(1-\gamma)/2}}{c_5}(1+\kappa)$$
$$\left.\times\ \frac{S_\nu[\mathrm{Jy}]}{f\ d[\mathrm{kpc}]\ \theta[\mathrm{arcmin}]^3}\nu[\mathrm{GHz}]^{(\gamma-1)/2}\right)^{2/(\gamma+5)},$$

where $m_e c^2 \approx 8.187\times 10^{-7}$ erg. We also have

$$E_B=\frac{\gamma+1}{4}E_{\mathrm{CR}},\quad E_{\min}=\frac{\gamma+5}{\gamma+1}E_B. \tag{13}$$

This result is the same as in BK05.

2.2. *A More General Formula for κ*

Let us start again with Equation (1):

$$\epsilon\approx\int_{p_{\mathrm{inj}}}^{\infty}4\pi k p^{-\gamma}\left(\sqrt{p^2c^2+m^2c^4}-mc^2\right)dp$$
$$=4\pi kc(mc)^{2-\gamma}\int_{\frac{p_{\mathrm{inj}}}{mc}}^{\infty}x^{-\gamma}\left(\sqrt{x^2+1}-1\right)dx,\quad x=\frac{p}{mc}$$
$$=4\pi kc(mc)^{2-\gamma}I\left(\frac{p_{\mathrm{inj}}}{mc}\right). \tag{14}$$

The integral $I(x)$ can be expressed through the Gauss hypergeometric function ${}_2F_1$ (for $\gamma > 2$),

$$I(x)=\frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}-\frac{x^{1-\gamma}\left(1-{}_2F_1\left(-\frac{1}{2},\frac{1-\gamma}{2},\frac{3-\gamma}{2};-x^2\right)\right)}{\gamma-1}, \tag{15}$$

but we will try to find a simpler approximation. First, note that

$$I(x)\approx\frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}-\frac{x^{3-\gamma}}{2(3-\gamma)}+\frac{x^{5-\gamma}}{8(5-\gamma)}-\cdots,\quad x\to 0, \tag{16}$$

$$I(x)\approx\frac{x^{2-\gamma}}{\gamma-2},\quad x\to\infty. \tag{17}$$

We can therefore try an approximation ($2<\gamma<3$)

$$I(x)_{\mathrm{approx}}=\frac{\frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}-\frac{x^{3-\gamma}}{2(3-\gamma)}+F(\gamma)x^{5-\gamma}}{1+F(\gamma)(\gamma-2)x^3}, \tag{18}$$

which has correct limits when $x\to 0$ and $x\to\infty$. We shall find $F(\gamma)$ by matching the condition $I(1)=I(1)_{\mathrm{approx}}$ as follows:

$$F(\gamma)=\frac{\frac{1}{2(3-\gamma)}-\frac{1-{}_2F_1\left(-\frac{1}{2},\frac{1-\gamma}{2},\frac{3-\gamma}{2};-1\right)}{\gamma-1}}{1-(\gamma-2)\left(\frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}-\frac{1-{}_2F_1\left(-\frac{1}{2},\frac{1-\gamma}{2},\frac{3-\gamma}{2};-1\right)}{\gamma-1}\right)}. \tag{19}$$

Since the last expression also involves the hypergeometric function, we found by trial and error an approximation

$$F(\gamma)_{\mathrm{approx}}=\frac{17}{1250}\frac{(2\gamma+1)\gamma}{(\gamma-2)(5-\gamma)}. \tag{20}$$

From now on we will assume $I(x)=I(x)_{\mathrm{approx}}$ and $F(\gamma)=F(\gamma)_{\mathrm{approx}}$ (the relative error is less than 3.5%).

The total CR energy density is then

$$\epsilon_{\mathrm{CR}}=\epsilon_e+\epsilon_{\mathrm{ion}}=K_e(m_ec^2)^{2-\gamma}I\left(\frac{p_{\mathrm{inj}}^e}{m_ec}\right)+\sum_i K_i(m_ic^2)^{2-\gamma}I\left(\frac{p_{\mathrm{inj}}^i}{m_ic}\right), \tag{21}$$

where (because $\frac{p_{\mathrm{inj}}^i}{m_ic}\ll 1$)

$$\epsilon_{\mathrm{ion}}\approx\sum_i K_i(m_ic^2)^{2-\gamma}\left(\frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}-\frac{1}{2(3-\gamma)}\left(\frac{\sqrt{E_{\mathrm{inj}}^2+2m_ic^2E_{\mathrm{inj}}}}{m_ic^2}\right)^{3-\gamma}\right)$$
$$\approx K_p(m_pc^2)^{2-\gamma}\sum_i\frac{n_i}{n_p}\left(\frac{p_{\mathrm{inj}}^i}{p_{\mathrm{inj}}^p}\right)^{\gamma-1}\left(\frac{m_i}{m_p}\right)^{2-\gamma}\times\left(\frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}-\frac{1}{2(3-\gamma)}\left(\frac{2E_{\mathrm{inj}}}{m_ic^2}\right)^{(3-\gamma)/2}\right)$$
$$\approx K_p(m_pc^2)^{2-\gamma}\sum_i\left[\frac{n_i}{n_p}\left(\frac{m_i}{m_p}\right)^{(3-\gamma)/2}\frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}-\frac{1}{2(3-\gamma)}\left(\frac{2E_{\mathrm{inj}}}{m_pc^2}\right)^{(3-\gamma)/2}\frac{n_i}{n_p}\right]$$
$$\approx K_p(m_pc^2)^{2-\gamma}\frac{n}{n_p}\left[\frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}\sum_i A_i^{(3-\gamma)/2}\nu_i-\frac{1}{2(3-\gamma)}\left(\frac{2E_{\mathrm{inj}}}{m_pc^2}\right)^{(3-\gamma)/2}\right]. \tag{22}$$

Finally,

$$\epsilon_{\rm CR} = K_e(m_ec^2)^{2-\gamma}\left[I\left(\frac{\sqrt{E_{\rm inj}^2+2m_ec^2E_{\rm inj}}}{m_ec^2}\right) + \frac{1}{\sum_i Z_i\nu_i}\left(\frac{m_p}{m_e}\right)^{2-\gamma}\left(\frac{2m_pc^2E_{\rm inj}}{E_{\rm inj}^2+2m_ec^2E_{\rm inj}}\right)^{(\gamma-1)/2} \times\left(\frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}\sum_i A_i^{(3-\gamma)/2}\nu_i - \frac{1}{2(3-\gamma)} \times\left(\frac{2E_{\rm inj}}{m_pc^2}\right)^{(3-\gamma)/2}\right)\right]$$

$$= K_e(m_ec^2)^{2-\gamma}\frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}(1+\kappa), \tag{23}$$

where

$$\kappa = I\left(\frac{\sqrt{E_{\rm inj}^2+2m_ec^2E_{\rm inj}}}{m_ec^2}\right)\left(\frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}\right)^{-1} + \frac{1}{\sum_i Z_i\nu_i}\left(\frac{m_p}{m_e}\right)^{2-\gamma}\left(\frac{2m_pc^2E_{\rm inj}}{E_{\rm inj}^2+2m_ec^2E_{\rm inj}}\right)^{(\gamma-1)/2} \times\left(\sum_i A_i^{(3-\gamma)/2}\nu_i - \frac{1}{2(3-\gamma)}\left(\frac{2E_{\rm inj}}{m_pc^2}\right)^{(3-\gamma)/2} \times\left(\frac{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma-2}{2}\right)}{2\sqrt{\pi}(\gamma-1)}\right)^{-1}\right) - 1. \tag{24}$$

In the above derivation, we use the fact that (Bell 1978b)

$$K_i/K_p = \frac{n_i}{n_p}\left(\frac{p^i_{\rm inj}}{p^p_{\rm inj}}\right)^{\gamma-1} \approx (n_i/n_p)(m_i/m_p)^{(\gamma-1)/2} \tag{25}$$

and

$$K_p/K_e = (n_p/n_e)\left(\frac{E_{\rm inj}^2+2m_pc^2E_{\rm inj}}{E_{\rm inj}^2+2m_ec^2E_{\rm inj}}\right)^{(\gamma-1)/2} \approx (n_p/n_e)\left(\frac{2m_pc^2E_{\rm inj}}{E_{\rm inj}^2+2m_ec^2E_{\rm inj}}\right)^{(\gamma-1)/2}. \tag{26}$$

Equation (24) has the correct limit (3) when $E_{\rm inj} \ll m_ec^2 \ll m_pc^2$. From Figure 1 it can be seen that for low $E_{\rm inj}$, the CR energy density is almost constant (independent of $E_{\rm inj}$) and use of Equation (3) is justified. When the shock velocity can be estimated, one should calculate the injection energy as $E_{\rm inj} \approx 4\frac{1}{2}m_pv_s^2$ and use Equation (24). Formulas (12) and (13) for magnetic field and minimum energy remain the same.[6] In Figure 2, we give the proton-to-electron energy density ratio as a function of injection energy in our approximation compared to the same data from Bell (1978b). The agreement is quite good despite the approximate character of our formulas.

[6] Note that κ is no longer the ions-to-electrons energy ratio but rather a suitable parameter introduced to make the new formulas the same as the old ones.

We have implemented our modified equipartition calculation by developing a PHP code.[7] The code uses some "typical" starting values for the radio spectral index, frequency, flux density, distance, angular radius, filling factor, shock velocity, and abundances, all of which can be changed or left as such. For example, if the shell thickness relative to the SNR radius δ can be measured, the volume filling factor is $f = 1 - (1-\delta)^3$. Otherwise a typical value $f = 0.25$ can be used (shell thickness of about 10%). If the shock velocity is unknown, one should leave 0 (and a simpler equipartition calculation can be performed by using Equation (3)). Simple ISM abundances are assumed initially (H:He ratio 10:1). In the implementation of our calculation, we used an approximation for the Gamma function (Nemes 2010):

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{1}{e}\left(z + \frac{1}{12z - \frac{1}{10z}}\right)\right)^z. \tag{27}$$

3. DISCUSSION

From the mathematical point of view, the equipartition calculation is the problem of solving a system of two independent equations (the synchrotron emissivity equation (4) and the equation for the total energy in a source (7)) for the three unknown variables (the total energy E, energy contained in the CRs $E_{\rm CR}$ (or K_e), and energy contained in the magnetic field E_B (or B)). This problem is, of course, impossible to solve without additional assumptions. The primary assumption is to seek the minimum of the total energy of the synchrotron source. Differentiation of Equation (8) ensures that the total energy disappears as an unknown variable, and the two starting equations (Equations (4) and (9)) can now give us solutions for both remaining unknown variables (K_e and B). As a result of differentiation of Equation (8), the exact equipartition between energies contained in the magnetic field and CRs is only approximately fulfilled (Equation (13)). The alternative assumption, commonly adopted, is the equipartition between energy contained in the CRs and in the magnetic field ($\epsilon_{\rm CR} = \epsilon_B$). By assuming this we directly link K_e and B (see Equations (2) and (7)). In the literature, these two calculations are commonly referred to as either equipartition or the minimum-energy calculation. Here, we would like to emphasize that strict equipartition does not have to be assumed to perform the calculation—if $\epsilon_B/\epsilon_{\rm CR} = \beta =$ constant is somehow known (independent information about CR electrons can come from X-ray data (inverse Compton effect) or about CRs from gamma rays (bremsstrahlung or pion decay)), the system can be solved. This means that the magnetic field energy density can be any constant fraction of the CR energy density, and the "equipartition" procedure will give appropriate formulas for the estimation of the amount of the total energy in a source and magnetic field strength, namely

$$B' = \left(\frac{4\beta}{\gamma+1}\right)^{2/(\gamma+5)}B, \tag{28}$$

where B' is the recalculated field for $\beta =$ constant, while B is the field corresponding to the minimum energy. The total energy calculated in this way is always higher than the minimal energy obtained from the equipartition, i.e., the minimum-energy calculation. However, the magnetic field can be either larger or smaller.

[7] The calculator is available at http://poincare.matf.bg.ac.rs/∼arbo/eqp/

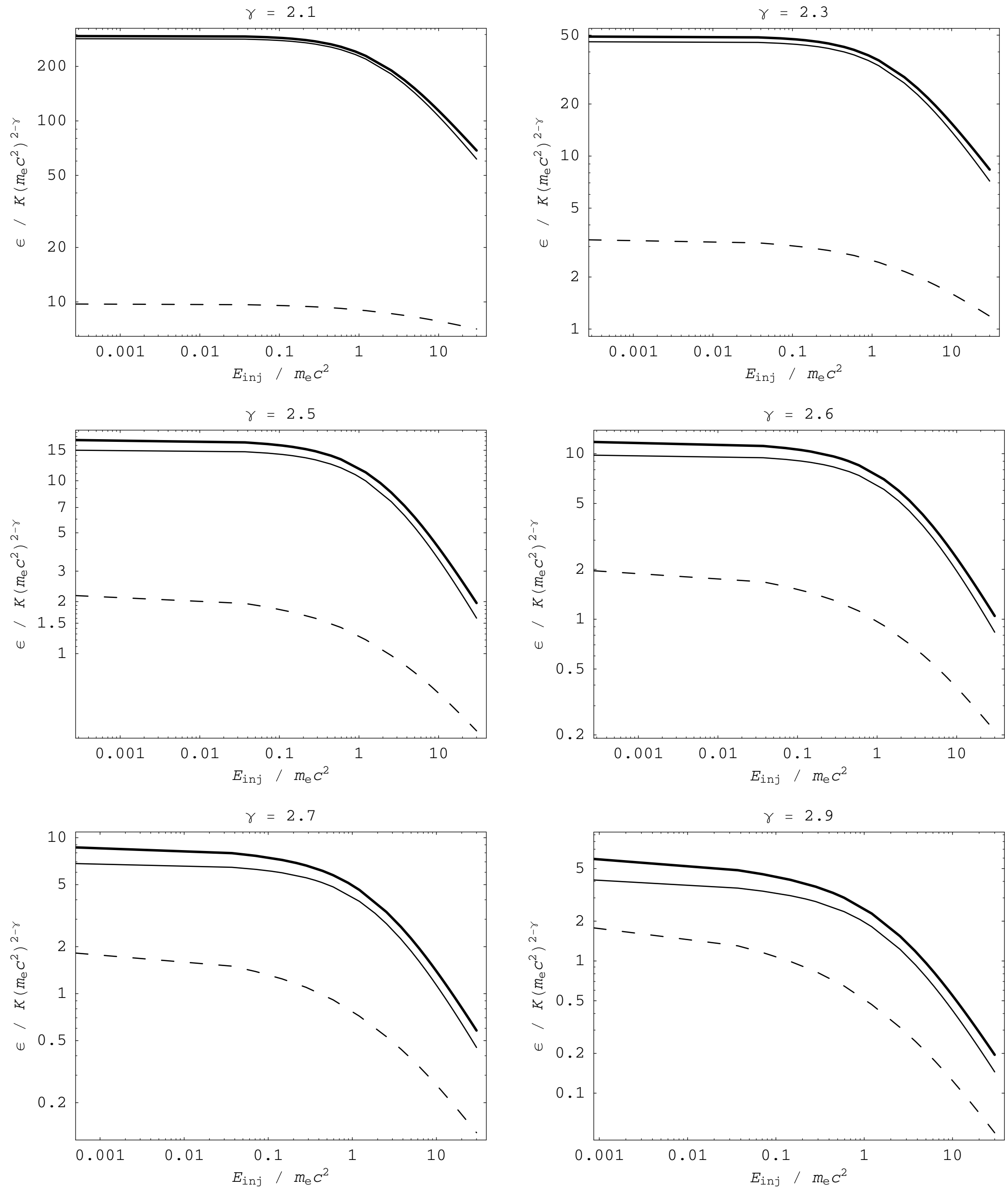

Figure 1. CR energy density of ions (H:He = 10:1, solid line), electrons (dashed line), and total (thick solid line) in our approximation as a function of injection energy.

Given the above, the equipartition calculation is not a precise method for determination of the magnetic field strength, but we can surely estimate its order of magnitude (Duric 1990). The main question is whether there is a physical relation between K_e and B. From Bell's (1978b) theory, K_e depends on the CR energy density $\epsilon_{\rm CR}$, injection energy $E_{\rm inj}$, and the energy spectral index of CR particles γ. Thus, implicitly, it must depend on the shock velocity, which itself depends on time t or radius $R = R(t)$ of an SNR. If there is evolution of the magnetic field $B = B(t)$, K_e and B must be related. Additionally, in the advanced model of DSA, a significant fraction of shock energy is transferred to CRs so the CR pressure has to be included in the equations (Drury 1983). From this, the so-called nonlinear DSA theory, strong magnetic field amplification (approximately two order of magnitudes) is expected especially in the early free-expansion phase of SNR evolution, when very strong shock waves exist (Bell 2004). The nonlinear effects thus produce efficient CR acceleration and, at the same time, significant amplification of the magnetic field strength. This increasing trend for both energy constituents of the synchrotron emission again leads to some form of non-strict equipartition.

The derivation procedure presented in Section 2.1, where integration limits for momenta are from 0 to ∞, leads to Equation (12). Using this equation and Equation (3), the calculated values of the magnetic field strength are slightly overestimated (by a few percent or more, depending on $E_{\rm inj}$). On the other hand, we neglect all kinds of energy losses in this paper. The main processes responsible for energy loss by the relativistic electrons are synchrotron radiation and inverse Compton scattering. These energy losses become significant for electrons especially at very high energies (the radiation power for both processes depends on the square of the electron

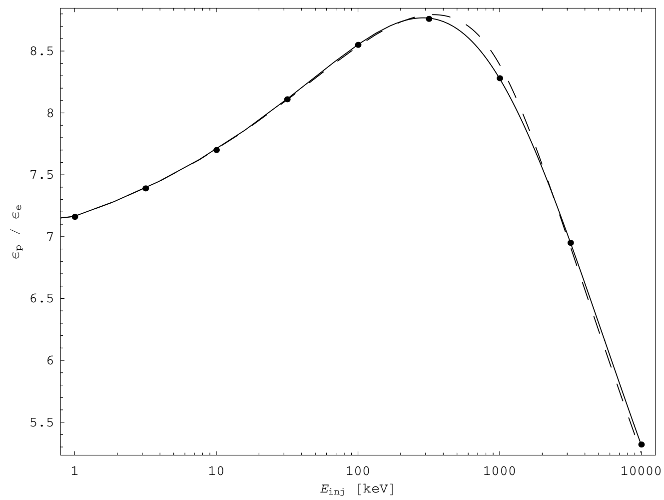

Figure 2. Proton-to-electron energy density ratio as a function of injection energy in our approximation (dashed line) and exact ratio (solid line) for $\gamma = 2.5$. Data points are from Bell (1978b).

kinetic energy). The energy losses of electrons result in an underestimation of the equipartition magnetic field strength. Thus, in our "simple approach" (Section 2.1), the effects of extending integration limits and energy losses work in opposite directions and may roughly cancel each other. If the integration limits are from $p_{\rm inj}$ to ∞ (Section 2.2), the equipartition calculation is derived correctly (without the assumption about the low shock velocity), but the problem of the energy losses remains and the equipartition estimates fail for electrons at the highest energies (BK05). This discussion is concentrated only on the energy losses of CR electrons. The energy pool of CRs is mainly filled with protons and heavier particles that do not lose energy heavily by synchrotron radiation and by inverse Compton scattering. Following Bell's (1978b) theory, the energy ratio between CR protons and electrons for the energy spectral index $\gamma = 2$ is approximately 40. If we take $\gamma = 2.5$, which is assumed for the curves presented in Figure 2, this ratio is $\approx$7. Due to this, the total CR energy losses can be neglected to a first approximation, especially for objects with harder spectra (SNRs) where the energy indices are lower.[8] However, the injection theory has been developed for protons and heavy particles, but not for electrons, which may or may not follow the protons. Hence, Bell's formula may give only lower limits for the proton-to-electron ratio at high energies and for the field strength.

In Table 1, we present values of the magnetic field strength and the minimal energy for the sample of 30 Galactic SNRs for which all data[9] necessary for the calculation can be found in the literature. The calculated magnetic field strengths are close to those calculated using revised equipartition (BK05) and are higher than those calculated using classical equipartition (P70) for all 30 SNRs (see Figures 3 and 4). For the P70 calculation we use $\mathcal{K} = (m_p/m_e)^{(3-\gamma)/2}$, $f = 0.25$, and a frequency interval 10^7 Hz $< \nu < 10^{11}$ Hz. For the BK05 calculation, in order to convert from specific intensity to flux density we use $I_\nu/l = L_\nu/4\pi V$ ($L_\nu = 4\pi d^2 S_\nu$),[10] $\mathbf{K_0} = (m_p/m_e)^\alpha$, and $f = 0.25$. For five younger Galactic SNRs, for which the forward shock velocities are known, we use a general equation for κ (Equation (24)). The differences between the calculated values, obtained using the general and simple approaches, are generally not so high. If we define the fractional error

$$\varphi = \frac{|B - B_{v_s=0}|}{B}, \tag{29}$$

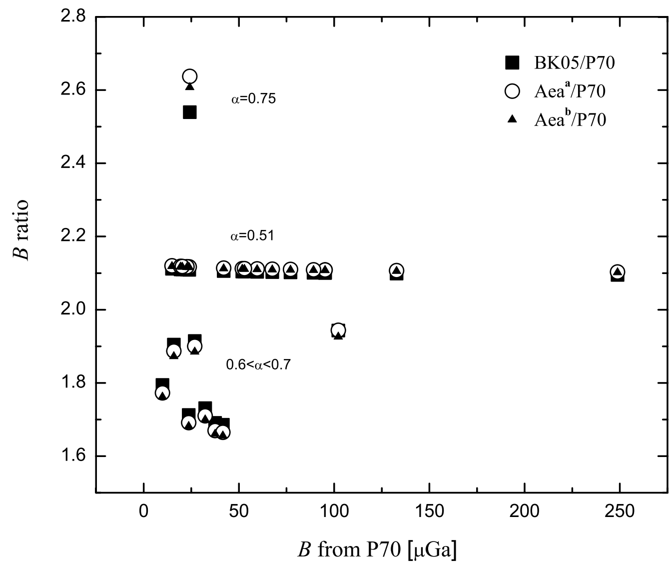

Figure 3. Comparison between different calculations for the minimum-energy magnetic field strength (B). "B ratio" represents the ratio between BK05 or this paper's calculations and the classical equipartition results (P70). Aea[a]—this paper, simple approach for p^+:$e^- = 1$:1; Aea[b]—this paper, simple approach for H:He = 10:1. Data are from Table 1 (25 SNRs).

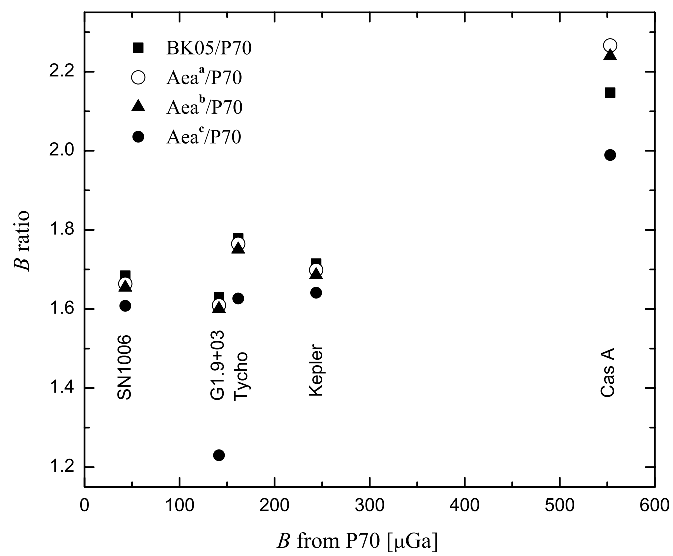

Figure 4. Comparison between different calculations for the minimum-energy magnetic field strength (B) for five young SNRs with available forward shock velocities. "B ratio" represents the ratio between BK05 or this paper's calculations and classical equipartition results (P70). Aea[a]—this paper, simple approach for p^+:$e^- = 1$:1; Aea[b]—this paper, simple approach for H:He = 10:1; Aea[c]—this paper, general approach for H:He = 10:1. Data are from Table 1 (five SNRs).

[8] The average energy index for SNRs is $\gamma \approx 2$ (the radio spectral index $\alpha \approx 0.5$).

[9] Including the distances to SNRs independent of the Σ–D relation (see Urošević et al. 2010 and references therein), and spectral indices $0.5 < \alpha < 1$.

[10] At the end of p. 415 of their paper, BK05 suggested replacing I_ν/l with L_ν/V. This is incorrect; 4π is missing in the denominator of the latter expression.

Table 1
Calculated Magnetic Field Strengths and Total Energies for a Sample of 30 Galactic SNRs

Name[a]	Other Names	Pacholczyk (1970)		Beck & Krause (2005)		This Paper[b]		This Paper[c]		This Paper[d]	
		B	E_{min}	B	E_{min}	B	E_{min}	B	E_{min}	B	E_{min}
G4.5+6.8[e]	Kepler, SN1604, 3C358	2.44E−04	8.38E+47	4.18E−04	2.34E+48	4.14E−04	2.30E+48	4.11E−04	2.26E+48	4.00E−04	2.14E+48
G21.80.6	Kes 69	7.71E−05	1.10E+50	1.62E−04	4.82E+50	1.63E−04	4.86E+50	1.63E−04	4.86E+50	–	–
G23.30.3	W41	6.75E−05	5.86E+49	1.42E−04	2.57E+50	1.43E−04	2.59E+50	1.42E−04	2.59E+50	–	–
G27.4+0.0	4C04.71	1.02E−04	3.75E+48	1.99E−04	1.32E+49	1.99E−04	1.33E+49	1.97E−04	1.30E+49	–	–
G33.6+0.1	Kes 79, 4C00.70, HC13	9.52E−05	3.79E+49	2.00E−04	1.66E+50	2.01E−04	1.68E+50	2.01E−04	1.67E+50	–	–
G46.80.3	HC30	5.96E−05	4.88E+49	1.25E−04	2.14E+50	1.26E−04	2.16E+50	1.26E−04	2.16E+50	–	–
G53.62.2	3C400.2, NRAO 611	2.42E−05	3.18E+48	6.14E−05	1.88E+49	6.38E−05	2.02E+49	6.30E−05	1.98E+49	–	–
G65.1+0.6	···	9.90E−06	1.90E+50	1.78E−05	5.87E+50	1.76E−05	5.73E+50	1.74E−05	5.66E+50	–	–
G93.70.2	CTB 104A, DA 551	2.68E−05	1.09E+49	5.13E−05	3.80E+49	5.09E−05	3.74E+49	5.05E−05	3.68E+49	–	–
G96.0+2.0	···	1.49E−05	2.20E+48	3.15E−05	9.74E+48	3.16E−05	9.82E+48	3.16E−05	9.81E+48	–	–
G108.20.6	···	1.94E−05	2.52E+49	4.09E−05	1.12E+50	4.11E−05	1.13E+50	4.11E−05	1.12E+50	–	–
G109.11.0	CTB 109	5.18E−05	1.40E+49	1.09E−04	6.16E+49	1.09E−04	6.21E+49	1.09E−04	6.20E+49	–	–
G111.72.1[f]	Cassiopeia A, 3C461	5.53E−04	1.32E+49	1.19E−03	5.56E+49	1.25E−03	6.19E+49	1.24E−03	6.05E+49	1.10E−03	4.76E+49
G114.3+0.3	···	2.40E−05	6.05E+47	5.05E−05	2.67E+48	5.07E−05	2.69E+48	5.07E−05	2.69E+48	–	–
G116.5+1.1	···	2.27E−05	6.21E+48	4.80E−05	2.75E+49	4.82E−05	2.77E+49	4.81E−05	2.76E+49	–	–
G116.9+0.2	CTB 1	3.23E−05	1.48E+48	5.60E−05	4.26E+48	5.53E−05	4.16E+48	5.49E−05	4.10E+48	–	–
G120.1+1.4[g]	Tycho, 3C10, SN1572	1.62E−04	1.63E+48	2.88E−04	4.88E+48	2.85E−04	4.80E+48	2.83E−04	4.73E+48	2.63E−04	4.09E+48
G132.7+1.3	HB3	2.36E−05	2.69E+49	4.05E−05	7.58E+49	4.00E−05	7.39E+49	3.98E−05	7.31E+49	–	–
G160.9+2.6	HB9	1.58E−05	3.08E+50	3.02E−05	1.06E+51	2.99E−05	1.04E+51	2.97E−05	1.03E+51	–	–
G205.5+0.5	Monoceros Nebula	2.03E−05	6.65E+49	4.27E−05	2.94E+50	4.29E−05	2.97E+50	4.29E−05	2.96E+50	–	–
G260.43.4	Puppis A, MSH 0844	5.29E−05	4.31E+49	1.11E−04	1.90E+50	1.12E−04	1.91E+50	1.12E−04	1.91E+50	–	–
G292.20.5	···	4.20E−05	4.78E+49	8.84E−05	2.11E+50	8.87E−05	2.12E+50	8.87E−05	2.12E+50	–	–
G296.80.3	115662	3.75E−05	6.50E+49	6.34E−05	1.41E+50	6.26E−05	1.38E+50	6.22E−05	1.36E+50	–	–
G304.6+0.1	Kes 17	9.52E−05	3.73E+49	2.00E−04	1.64E+50	2.01E−04	1.65E+50	2.01E−04	1.65E+50	–	–
G315.42.3	RCW 86, MSH 1463	4.16E−05	1.37E+49	7.01E−05	3.75E+49	6.92E−05	3.66E+49	6.88E−05	3.62E+49	–	–
G327.6+14.6[h]	SN1006, PKS 145941	4.28E−05	4.65E+48	7.22E−05	1.27E+49	7.13E−05	1.24E+49	7.09E−05	1.22E+49	6.89E−05	1.16E+49
G332.40.4	RCW 103	1.33E−04	4.63E+48	2.79E−04	2.03E+49	2.80E−04	2.04E+49	2.80E−04	2.04E+49	–	–
G337.80.1	Kes 41	8.92E−05	6.80E+49	1.87E−04	2.99E+50	1.88E−04	3.01E+50	1.88E−04	3.00E+50	–	–
G349.7+0.2	···	2.49E−04	6.49E+49	5.21E−04	2.83E+50	5.23E−04	2.85E+50	5.23E−04	2.85E+50	–	–
G1.9+03[i]	···	1.41E−04	3.65E+47	2.30E−04	9.33E+47	2.28E−04	9.11E+47	2.26E−04	9.00E+47	1.74E−04	5.31E+47

Notes. All units are in the cgs system. B is the magnetic field strength calculated for the minimum-energy assumption.

[a] According to Green's (2009) catalog, from which data for SNRs, except shock velocities, have been taken.

[b] Simple approach for $p^+{:}e^- = 1{:}1$.

[c] Simple approach for H:He = 10:1.

[d] General approach for H:He = 10:1, for young SNRs with available forward shock velocities (υ_s).

[e] $\upsilon_s = 1660\ \mathrm{km\ s^{-1}}$ (Sankrit et al. 2005).

[f] $\upsilon_s = 4900\ \mathrm{km\ s^{-1}}$ (Patnaude & Fesen 2009).

[g] $\upsilon_s = 4700\ \mathrm{km\ s^{-1}}$ (Hayato et al. 2010).

[h] $\upsilon_s = 2890\ \mathrm{km\ s^{-1}}$ (Ghavamian et al. 2002).

[i] $\upsilon_s = 14000\ \mathrm{km\ s^{-1}}$ (Carlton et al. 2011).

$\bar{\varphi} = 11\%$ for the five SNRs with estimated shock velocities. For the youngest Galactic SNR G1.9+0.3, the fractional error is the largest, $\varphi_{\max} = 30\%$ (see Figure 4). Further inspection of Table 1 and Figures 3 and 4 leads to the conclusion that the variation in abundances of CR species does not significantly alter the final equipartition results.

4. CONCLUSIONS

In this paper we derived modified equipartition, i.e., the minimum-energy formula for estimating magnetic fields in SNRs. Our approach is similar to that of BK05 in the sense that we do not integrate over frequencies as P70 does; however, the following applies to our paper.

1. We assume power-law spectra $n(p) \propto p^{-\gamma}$ and integrate over momentum to obtain energy densities of particles.
2. We take into account different ion species and not only an equal number of protons and electrons at injection (e.g., for an H to He ratio of 10:1 there is more energy in α-particles than in electrons).
3. We use flux density at a given frequency and also assume an isotropic distribution of the pitch angles for the remnant as a whole.
4. By incorporating the dependence $\epsilon = \epsilon(E_{\rm inj})$ we make the formula applicable to younger remnants as well.
5. We calculate the magnetic field strengths for a sample of 30 Galactic SNRs and obtain values that are close to those calculated by using revised equipartition (BK05) and higher than those calculated by using classical equipartition (P70).

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