Formalizing Analytic Geometries

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Abstract. We present our current work on formalizing analytic (Cartesian) plane geometries within the proof assistant Isabelle/HOL. We give several equivalent definitions of the Cartesian plane and show that it models synthetic plane geometries (using both Tarski's and Hilbert's axiom systems). We also discuss several techniques used to simplify and automate the proofs. As one of our aims is to advocate the use of proof assistants in mathematical education, our exposure tries to remain simple and close to standard textbook definitions. Our other aim is to develop the necessary infrastructure for implementing decision procedures based on analytic geometry within proof assistants.

1 Introduction

In classic mathematics, there are many different geometries. Also, there are different viewpoints on what is considered to be standard (Euclidean) geometry. Sometimes, geometry is defined as an independent formal theory, sometimes as a specific model. Of course, the connections between different foundations of geometry are strong. For example, it can be shown that the Cartesian plane represents (a canonical) model of formal theories of geometry.

The traditional Euclidean (synthetic) geometry, dating from the ancient Greece, is a geometry based on a, typically small, set of primitive notions (e.g., points, lines, congruence relation, ...) and axioms (implicitly defining these primitive notions). There is a number of variants of axiom systems for Euclidean geometry and the most influential and important ones are Euclid's system (from his seminal "Elements") and its modern reincarnations [1], Hilbert's system [9], and Tarski's system [21].

One of the most influential inventions in mathematics, dating from the XVII century, was the Descartes's invention of coordinate system, allowing algebraic equations to be expressed as geometric shapes. The resulting *analytic (or Cartesian) geometry* bridged the gap between algebra and geometry, crucial to the discovery of infinitesimal calculus and analysis.

With the appearance of modern proof assistants, in recent years, many classical mathematical theories have been formally analyzed mechanically, within proof assistants. This has also been the case with geometry and there have been several attempts to formalize different geometries and different approaches to geometry. We are not aware that there have been full formalizations of the seminal Hilbert's [9] or Tarski's [21] development, but significant steps have been made and major parts of these theories have been formalized within different proof assistants. As the common experience shows, using the proof assistants significantly raises the level of rigour as many classic textbook developments turn out to be imprecise or sometimes even flawed. Therefore, any formal treatment of geometry, including ours, should rely on using proof assistants. Therefore, all the work presented in this paper is done within Isabelle/HOL proof assistant [17][1].

Main applications of our present work are in automated theorem proving in geometry and in mathematical education and teaching of geometry.

When it comes to automated theorem proving in geometry (GATP), the analytic approach has shown to be superior . The most successful methods in this field are *algebraic methods* (e.g., Wu's method [22] and the Gröbner bases method [2, 11]) relying on the coordinate representation of points. Modern theorem provers relying on these methods have been used to show hundreds of non-trivial theorems. On the other hand, theorem provers based on synthetic axiomatizations have not been so successfull. Most GATP systems are used as trusted software tools as they are usually not connected to modern proof assistants. In order to increase their reliability, they should be connected to the modern proof assistants (either by implementing them and proving their correctness within proof assistants, or by having proof assistants check their claims). Several steps in this direction have already been made [7, 14].

In mathematics education in high-schools and in entry levels of university both approaches (synthetic and analytic) to geometry are usually demonstrated. However, while the synthetic approach is usually taught in its full rigor (aiming to serve as an example of rigorous axiomatic development), the analytic geometry is usually presented much more informally (sometimes just as a part of calculus). Also, these two approaches are usually presented independently, and the connections between the two are rarely formally proved within a standard curriculum.

Having this in mind, this work tries to bridge several gaps that we feel are present in current state-of-the-art in the field of formalizations of geometry.

1. First, we aim to formalize Cartesian geometry within a proof assistant, in a rigorous manner, but still very close to standard high-school exposures.
2. We aim to show that several different definitions of basic notions of analytic geometry found in various textbooks all turn out to be equivalent, therefore representing a single abstract entity — the Cartesian plane.
3. We aim to show that the standard Cartesian plane geometry represents a model of several geometry axiomatizations (most notably Tarski's and Hilbert's).
4. We want to formally analyze model-theoretic properties of different axiomatic systems (for example, we want to show that all models of Hilbert's geometry are isomorphic to the standard Cartesian plane).
5. We want to formally analyze axiomatizations and models of non-Euclidean geometries and their properties (e.g., to show that the Poincaré disk is a model of the Lobachevsky's geometry).

[1] Proof documents are available online at http://argo.matf.bg.ac.rs

6. We want to formally establish connections of the Cartesian plane geometry with algebraic methods that are the most successfull methods in GATP.

Several of these aims have been already established, while some other are still in progress. In this paper we will describe the first three points. The last point has already been discussed in [14], while other points are left for further work.

Apart from having many theorems formalized and proved within Isabelle/HOL, we also discuss our experience in applying different techniques used to simplify the proofs. The most significant was the use of "without the loss of generality (wlog)" reasoning, following the approach of Harrison [8] and justified by using various isometric transformations.

Overview of the paper. In Section 2 some background on Isabelle/HOL and the notation used is given. In Section 3 we give several definitions of basic notions of the Cartesian plane geometry and prove their equivalence. In Section 4 we discuss the wlog reasoning and the use of isometric transformations in formal geometry proofs. In Section 5 and Section 6 we show that our Cartesian plane geometry models the axioms of Tarski and the axioms of Hilbert. In Section 7 we discuss the current state-of-the-art in formalizations of geometry. Finally, in Section 8 we draw some conclusions and discuss future work.

2 Background

Isabelle/HOL.

3 Formalizing Cartesian Geometry

When formalizing a theory, one should decide which notions are considered to be primitive, and which notions are defined, based on those primitive notions. Our formalization of analytic geometry aims at establishing the connection with synthetic geometries so it follows primitive notions given in the synthetic approach. Each geometry considers a class of objects called the *points*. Richer geometries, such as Hilbert's also consider distinct set of objects called the *lines*, while Tarski's geometry does not consider lines, at all. In some expositions of geometry, lines are a defined notion, and they are defined as sets of points. This assumes dealing with the full set theory, and many axiomatizations try to avoid this. So, we are going to define both points and lines, since we want to allow to analyzing both Tarski's and Hilbert's geometry. The basic relation connecting points and lines is *incidence*, informally stating that a line contains a point (or dually that the point is contained in a line). Other primitive relations (in most axiomatic systems) are *betweenness*, defining the order of collinear points, and *congruence*.

3.1 Points in Analytic Geometry.

Point in a real Cartesian plane is determined by its x and y coordinate. So, points are pairs of real numbers ($\mathbb{R}^2$), what can be easily formalized in Isabelle/HOL by `type_synonym` $point^{ag}$ = "$real \times real$".

3.2 The Order of Points.

The order of (collinear) points is defined using the *betweenness* relation. This is a ternary relation and $\mathcal{B}(A, B, C)$ denotes that points A, B, and C are collinear and that B is between A and C. However, some axiomatizations (e.g., Tarski's) allow the case when B is equal to A or C (we will say the between relation is inclusive), while some other (e.g., Hilbert's) do not (and we will say that the between relation is exclusive). In the first case, the between relation holds if there is a real number $0 \leq k \leq 1$ such that $\overrightarrow{AB} = k \cdot \overrightarrow{AC}$. We want to avoid explicitly defining vectors (as they are usually not a primitive, but a derived notion in synthetic geometries) and so we formalized betweenness in Isabelle/HOL as following:

$$\begin{aligned}&\mathcal{B}_T^{ag}\ (xa, ya)\ (xb, yb)\ (xc, yc) \longleftrightarrow\\ &\quad (\exists (k :: real).\ 0 \leq k\ \wedge\ k \leq 1\ \wedge\\ &\qquad (xb - xa) = k \cdot (xc - xa)\ \wedge\ (yb - ya) = k \cdot (yc - ya))\end{aligned}$$

If A, B, and C are required to be distinct, then $0 < k < 1$ must hold, and the relation is denoted by $\mathcal{B}_H^{ag}$.

3.3 Congruence.

The congruence relation is defined on pairs of points. Informally, $AB \cong_t CD$ denotes that the segment AB is congruent to the segment CD. Standard metric in $\mathbb{R}^2$ defines that distance of points $A(x_A, y_A)$, $B(x_B, y_B)$ to be $d(A, B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$. Squared distance is defined as $d^{2,ag}\ A\ B = (x_B - x_A)^2 + (y_B - y_A)^2$. The points A, B are congruent to the points C, D iff $d^{2,ag}\ A\ B = d^{2,ag}\ C\ D$. In Isabelle/HOL this can be formalized as:

$$\begin{aligned}&d^{2,ag}\ (x_1, y_1)\ (x_2, y_2) = (x_2 - x_1) \cdot (x_2 - x_1) + (y_2 - y_1) \cdot (y_2 - y_1)\\ &A_1B_1 \cong^{ag} A_2B_2 \longleftrightarrow d^{2,ag}\ A_1\ B_1 = d^{2,ag}\ A_2\ B_2\end{aligned}$$

3.4 Lines and incidence.

Line equations. Lines in the Cartesian plane are usually represented by the equations of the form $Ax + By + C = 0$, so a triplet $(A, B, C) \in \mathbb{R}^3$ determines a line. However, triplets where $A = 0$ and $B = 0$ do not correspond to valid equations and must be excluded. However, equations $Ax + By + C = 0$ and $kAx + kBy + kC = 0$, for a real $k \neq 0$, define a same line. So, a line must not be defined just by a single equation, but a line must be defined as a class of equations that have proportional coefficients. Formalization in Isabelle/HOL proceeds in several steps. First, the domain of valid equation coefficients (triplets) is defined.

```
typedef line_coeffs^ag =
  {((A :: real), (B :: real), (C :: real)). A ≠ 0 ∨ B ≠ 0}
```

When this type is defined, the function *Rep_line_coeffs* converts abstract values of this type to their concrete underlying representations (tripplets of reals), and the function *Abs_line_coeffs* converts (valid) tripplets to abstract values of this type.

Two tripplets are equivalent iff they are proportional.

$$
\begin{aligned}
&l_1 \approx^{ag} l_2 \longleftrightarrow \\
&\quad (\exists\ A_1\, B_1\, C_1\, A_2\, B_2\, C_2. \\
&\quad (\mathit{Rep\_line\_coeffs}\ \ l_1 = (A_1, B_1, C_1)) \ \wedge\ \mathit{Rep\_line\_coeffs}\ \ l_2 = (A_2, B_2, C_2)\ \wedge \\
&\quad (\exists k.\ \ k \neq 0\ \wedge\ A_2 = k \cdot A_1\ \wedge\ B_2 = k \cdot B_1\ \wedge\ C_2 = k \cdot C_1))
\end{aligned}
$$

It is shown that this is indeed an equivalence relation. The definition of the type of lines uses the support for quotient types and quotient definitions that has been recently introduced to Isabelle/HOL [10]. So, lines (the type $line^{ag}$) are defined using the `quotient_type` command, as equivalence classes of the $\approx^{ag}$ relation.

To avoid using set theory, geometry axiomatizations that explicitly consider lines use the incidence relation. If the previous definition of lines is used, then checking incidence reduces to calculating whether the point (x, y) satisfies the line equation $A \cdot x + B \cdot y + C = 0$, for some representative coefficients A, B, and C. However, to show that this relation is well defined, it must be shown that if other representatives A', B', and C' are chosen (that are proportional to A, B, and C), then $A' \cdot x + B' \cdot y + C = 0$. In our Isabelle/HOL formalization, we use the quotient package. So,

$$
\begin{aligned}
&\mathit{ag\_in\_h}\ \ (x, y)\ \ l \longleftrightarrow \\
&\quad (\exists\ A\ \ B\ \ C.\ \ \mathit{Rep\_line\_coeffs}\ \ l = (A,\ \ B,\ \ C) \wedge (A \cdot x + B \cdot y + C = 0))
\end{aligned}
$$

Then, $A \in^{ag}_{H} l$ is defined using the `quotient_definition` based on the relation *ag_in_h*. The well-definedness lemma is

```
lemma
  assumes "l ≈ l'"
  shows "ag_in_h P l = ag_in_h P l'"
```

Affine definition. In affine geometry, a line is defined by fixing a point and a vector. As points, vectors also can be represented by pairs of reals `type_synonym` vec^{ag} = "$real \times real$". Vectors defined like this form vector space (with naturally defined vector addition and scalar multiplication). Points and vectors can be added as $(x, y) + (v_x, v_y) = (x + v_x, y + v_y)$. Then, line is represented by a Point and a non-zero vector:

```
typedef line_point_vec^ag = {(p :: point^ag, v :: vec^ag). v ≠ (0, 0)}
```

However, different points and vectors can determine a single line, and a quotient construction must be used again. The

$$l_1 \approx^{ag} l_2 \longleftrightarrow (\exists\, p_1\, v_1\, p_2\, v_2.$$
$$\quad Rep\_line\_point\_vec\ \ l_1 = (p_1, v_1) \wedge Rep\_line\_point\_vec\ \ l_2 = (p_2, v_2) \wedge$$
$$\quad (\exists k.\ \ v_1 = k \cdot v_2 \wedge p_2 = p_1 + k \cdot v_1))$$

It is shown that this is indeed an equivalence relation. Then, the type of lines ($line^{ag}$) is again defined by a quotient definitions) are defined using the `quotient_type` command, as equivalence classes of the $\approx^{ag}$ relation.

3.5 Isometries.

Isometries are usually defined notions in synthetic geometries. Reflections can be defined first, and then other isometries can be defined as compositions of reflections. However, in our current formalizations, isometries are used only as an auxiliary tool used to simplify our proofs (as discussed in Section 4). So we were not concerned with defining isometries in terms of primitive notions (points and congruence) but we give their separate (analytic) definitions and prove the properties needed in our later proofs.

Translation is defined for a given vector (not explicitly defined, but represented by a pair of reals). The formal definition in Isabelle/HOL is straightforward.

$$transp^{ag}\ \ (v_1, v_2)\ \ (x_1, x_2) = (v_1 + x_1, v_2 + x_2)$$

Rotation is parametrized for a real parameter α (representing the rotation angle), while only rotations around the origin are considered (other rotations can be obtained by composing translations and a rotation around the origin). Elementary trigonometry is used to give the following formal definition in Isabelle/HOL.

$$rotp^{ag}\ \ \alpha\ \ (x, y) = ((\cos\alpha) \cdot x - (\sin\alpha) \cdot y, (\sin\alpha) \cdot x + (\cos\alpha) \cdot y)$$

There is also central symmetry that is easily defined using point coordinates:

$$symp^{ag}\ \ (x, y) = (-x, -y)$$

Important properties of all isometries are invariance properties, i.e., they preserve basic relations (betweenness and congruency).

$$\mathcal{B}_T^{ag}\ \ A\ \ B\ \ C \longleftrightarrow \mathcal{B}_T^{ag}\ \ (transp^{ag}\ \ v\ \ A)\ \ (transp^{ag}\ \ v\ \ B)\ \ (transp^{ag}\ \ v\ \ C)$$
$$AB \cong^{ag} CD \longleftrightarrow$$
$$\quad (transp^{ag}\ \ v\ \ A)(transp^{ag}\ \ v\ \ B) \cong^{ag} (transp^{ag}\ \ v\ \ C)(transp^{ag}\ \ v\ \ D)$$
$$\mathcal{B}_T^{ag}\ \ A\ \ B\ \ C \longleftrightarrow \mathcal{B}_T^{ag}\ \ (rotp^{ag}\ \ \alpha\ \ A)\ \ (rotp^{ag}\ \ \alpha\ \ B)\ \ (rotp^{ag}\ \ \alpha\ \ C)$$
$$AB \cong^{ag} CD \longleftrightarrow (rotp^{ag}\ \ \alpha\ \ A)(rotp^{ag}\ \ \alpha\ \ B) \cong^{ag} (rotp^{ag}\ \ \alpha\ \ C)(rotp^{ag}\ \ \alpha\ \ D)$$

lemma `ag_symp_bet:`
$$\quad \mathcal{B}_T^{ag}\ \ A\ \ B\ \ C \longleftrightarrow \mathcal{B}_T^{ag}\ \ (symp^{ag}\ \ A)\ \ (symp^{ag}\ \ B)\ \ (symp^{ag}\ \ C)$$

lemma ag_symp_cong:
$AB \cong^{ag} CD \longleftrightarrow (symp^{ag}\ \ A)(symp^{ag}\ \ B) \cong^{ag} (symp^{ag}\ \ C)(symp^{ag}\ \ D)$

Isometries are used only to transform points to canonical position (usually to move them to the y-axis). The following lemmas show that this is possible.

$\exists v.\ \ transp^{ag}\ \ v\ \ P = (0,0)$
$\exists \alpha.\ \ rotp^{ag}\ \ \alpha\ \ P = (0,p)$
$\mathcal{B}_T^{ag}\ \ (0,0)\ \ P_1\ \ P_2 \longrightarrow \exists \alpha\ \ p_1\ \ p_2.\ \ rotp^{ag}\ \ \alpha\ \ P_1 = (0,p_1) \wedge rotp^{ag}\ \ \alpha\ \ P_2 = (0,p_2)$

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4 Using Isometric Transformations

One of the most important techniques used to simplify our formalization relied on using isometric transformations. We shall try to give a motivation for applying isometries on the following, simple example.

Let us prove that in our model, if $\mathcal{B}_T^{ag}\ A\ X\ B$ and $\mathcal{B}_T^{ag}\ A\ B\ Y$ then $\mathcal{B}_T^{ag}\ X\ B\ Y$. Even on this simple example, if a straightforward approach is taken and isometric transformations are not used the algebraic calculations become tedious.

Let $A = (x_A, y_A)$, $B = (x_B, y_B)$, and $X = (x_X, y_X)$. Since $\mathcal{B}_T^{ag}\ A\ X\ B$ holds, there is a real number k_1, $0 \le k_1 \le 1$, such that $(x_X - x_A) = k_1 \cdot (x_B - x_A)$, and $(y_X - y_A) = k_1 \cdot (y_B - y_A)$. Similarly, since $\mathcal{B}_T^{ag}\ A\ B\ Y$ holds, there is a real number k_2, $0 \le k_2 \le 1$, such that $(x_B - x_A) = k_2 \cdot (x_Y - x_A)$, and $(y_B - y_A) = k_2 \cdot (y_Y - y_A)$. Then, we can define a real number k by

$$\frac{k_2 - k_2 \cdot k_1}{1 - k_2 \cdot k_1}.$$

If $X \neq B$, then, using straightforward but complex algebraic calculations, it can be shown that $0 \le k \le 1$, and that $(x_B - x_X) = k \cdot (x_Y - x_X)$, and $(y_B - y_X) = k \cdot (y_Y - y_X)$, and therefore $\mathcal{B}_T^{ag}\ X\ B\ Y$ holds. The degenerate case $X = B$ holds trivially.

However, if we apply the isometric transformations, then we can assume that $A = (0,0)$, $B = (0, y_B)$, and $X = (0, y_X)$, and that $0 \le y_X \le y_B$. The case $y_B = 0$ holds trivially. Otherwise, $x_Y = 0$ and $0 \le y_B \le y_Y$. Hence $y_X \le y_B \le y_Y$, and the case holds. Note that in this case no significant algebraic calculations were needed and the proof relied only on simple transitivity properties of $\le$.

Comparing the previous two proofs, indicates how applying isometric transformations significantly simplifies the calculations involved and shortens the proofs.

Since this technique is used throughout our formalization, it is worth discussing what is the best way to formulate the appropriate lemmas that justify its use and use as much automation as possible. We followed the approach of Harrison [8].

The property P is invariant under the transformation t iff it is not affected after transforming the points by t.

$inv\ \ P\ \ t \longleftrightarrow (\forall\ \ A\ \ B\ \ C.\ \ P\ \ A\ \ B\ \ C \longleftrightarrow P\ \ (tA)\ \ (tB)\ \ (tC))$

Then, the following lemma can be used to reduce the statement to any three collinear points to the positive part of the y-axis (alternatively, x-axis could be chosen).

```
lemma
  assumes "∀ y_B y_C. 0 ≤ y_B ∧ y_B ≤ y_C ⟶ P (0,0) (0,y_B) (0,y_C)"
          "∀v. inv P (transp^ag v )" "∀α. inv P (rotp^ag α )"
          "inv P (symp^ag )"
  shows   "∀ A B C. B_T^ag A B C ⟶ P A B C"
```

It turns out that showing that the statement is invariant to isometric transformations is mostly done by automation using the lemmas stating that the betweenness and congruent relations are invariant to isometric transformations.

5 Tarski's geometry

Our goal in this section is to prove that our definitions of the Cartesian plane satisfy all the axioms of Tarski's geometry. Tarski's geometry considers only points, betweenness (denoted by $\mathcal{B}_t(A, B, C)$) and congruence (denoted by $AB \cong_t C$) as basic objects. In Tarski's geometry lines are not explicitly present and collinearity is defined by using the betweenness relation.

$\mathcal{C}_t(A, B, C) \longleftrightarrow \mathcal{B}_t(A, B, C) \vee \mathcal{B}_t(B, C, A) \vee \mathcal{B}_t(C, A, B)$

5.1 Axioms of congruence.

First three Tarski's axioms express basic properties of congruence.

$AB \cong_t BA$
$AB \cong_t CC \rightarrow\ \ A = B$
$AB \cong_t CD \wedge\ \ AB \cong_t EF\ \ \longrightarrow CD \cong_t EF$

We want to prove that our relation $\cong^{ag}$ satisfies the properties abstractly given by the previous axioms (i.e., that the given axioms hold for our Cartesian model). For example, for the first axiom this reduces to showing that $AB \cong^{ag} BA$. The proofs are rather straightforward and are done almost automatically (by simplifications after unfolding the definitions).

5.2 Axioms of Betweenness.

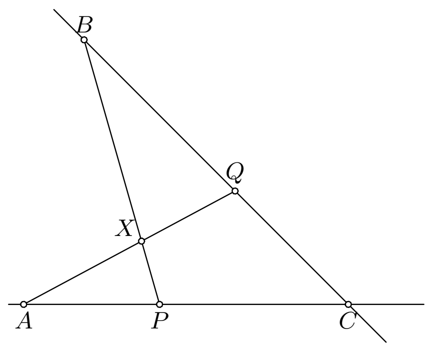
Identity of Betweenness. First axiom of (inclusive) betweenness gives its one simple property and, for our model, it is also proved almost automatically.

$\mathcal{B}_t(A, B, A) \rightarrow A = B$

The axiom of Pasch. The next axiom is the Pasch's axiom:

$\mathcal{B}_t(A, P, C) \wedge \mathcal{B}_t(B, Q, C) \rightarrow (\exists X.\ (\mathcal{B}_t(P, X, B) \wedge \mathcal{B}_t(Q, X, A)))$

Under the assumption that all points involved are distinct the picture corresponding to this axioms is:



Before we discuss the proof that our Cartesian plane satisfies this axiom we discuss some issues related to the Tarski's geometry that turned out to be important for our overall proof organization. The final version of Tarski's axiom system was designed to be minimal (it contains only 11 axioms), and the central axioms that describe the betweenness relation are the identity of betweenness and Pasch's axiom. In formalizations of Tarski's geometry ([16]), all other elementary properties of this relation are derived from these two axioms. For example, to derive the symmetry property (i.e., $\mathcal{B}_t(A, B, C) \Rightarrow \mathcal{B}_t(C, B, A)$), the axiom of Pasch is applied to triplets ABC and BCC and the point X is obtained so that $\mathcal{B}_t(C, X, A)$ and $\mathcal{B}_t(B, X, B)$, and then, by axiom 1, $X = B$ and $\mathcal{B}_t(C, B, A)$ holds. However, in our experience, in order to prove that our Cartesian plane models Tarski's axioms (especially the axiom of Pasch), it would be convenient to have some of its consequences (e.g., the symmetry and transitivity properties) already proved. Indeed, earlier variants of Tarski's axiom system contained more axioms, and these properties were separate axioms. From a more abstract viewpoint, the symmetry property seems to be simpler property then Pasch's axiom (for example, it involves only the points lying on a line, while the axiom of Pasch allows points that lie in a plane that are not necessarily collinear). Moreover, the previous proof uses rather subtle properties of the way that the Pasch's axiom is formulated. For example, if its conclusion used $\mathcal{B}_t(B, X, P)$ and $\mathcal{B}_t(A, X, Q)$ instead of $\mathcal{B}_t(P, X, B)$ and $\mathcal{B}_t(Q, X, A)$, then the proof could not be conducted. Therefore, we decided that a good approach would be to directly show that some elementary properties (e.g., symmetry, transitivity) of the betweenness relation hold in the model, and use these facts in the proof of much more complex Pasch's axiom.

$\mathcal{B}_T^{ag}\ A\ A\ B$
$\mathcal{B}_T^{ag}\ A\ B\ C \rightarrow \mathcal{B}_T^{ag}\ C\ B\ A$

$\mathcal{B}_T^{ag}\ A\ X\ B\ \wedge\ \mathcal{B}_T^{ag}\ A\ B\ Y \rightarrow \mathcal{B}_T^{ag}\ X\ B\ Y$
$\mathcal{B}_T^{ag}\ A\ X\ B\ \wedge\ \mathcal{B}_T^{ag}\ A\ B\ Y \rightarrow \mathcal{B}_T^{ag}\ A\ X\ Y$

Returning to the proof that our Cartesian plane satisfy the full Pasch's axiom, first several degenerate cases need to be considered. First group of degenerate cases arise when some points in the construction are equal. For example, $\mathcal{B}_t(A, P, C)$ allows that $A = P = C$, that $A = P \neq C$, that $A \neq P = C$ and that $A \neq P \neq C$. A direct approach would be to analyze all these cases separately. However, a better approach is to carefully analyze the conjecture and identify which cases are substantially different. It turns out that only two different cases are relevant. If $P = C$, then Q is the point sought. If $Q = C$, then P is the point sought. Next group of degenerate cases arise when all points are collinear. In this case, either $\mathcal{B}_t(A, B, C)$ or $\mathcal{B}_t(B, A, C)$ or $\mathcal{B}_t(B, C, A)$ holds. In the first case B is the point sought, in the second case it is the point A, and in the third case it is the point P.

Note that all degenerate cases that arise in the Pasch's axioms were proved directly by using these elementary properties and that coordinate computations did not need to be used in those cases. This suggests that degenerate cases of Pasch's axiom are equivalent to the conjunction of the given properties. Further, this suggests that if Tarski's axiomatics was changed so that it included these elementary properties, then the Pasch's axiom could be weakened so that it includes only the central case of non-collinear, distinct points.

Finally, the central case remains. In that case, algebraic calculations are used to calculate the coordinates of the point X and prove the conjecture. To simplify the proof, isometries are used, as described in Section 4. The configuration is transformed so that A becomes the origin $(0, 0)$, and so that $P = (0, y_P)$ and $C = (0, y_C)$ lie on the positive part of the y-axis. Let $B = (x_B, y_B)$, $Q = (x_Q, y_Q)$ and $X = (x_X, y_X)$. Since $\mathcal{B}_t(A, P, C)$ holds, there is a real number k_3, $0 \leq k_3 \leq 1$, such that $y_P = k_3 \cdot y_C$. Similarly, since $\mathcal{B}_t(B, Q, C)$ holds, there is a real number k_4, $0 \leq k_4 \leq 1$, such that $(x_B - x_A) = k_2 \cdot (x_Y - x_A)$, and $x_Q - x_B = -k_4 * x_B$ and $y_Q - y_B = k_4 * (y_C - y_B)$. Then, we can define a real number k_1 by

$$\frac{k_3 \cdot (1 - k_4)}{k_4 + k_3 - k_3 \cdot k_4}.$$

$A \neq P \neq C$ and points are not colinear, then, using straightforward algebraic calculations, it can be shown that $0 \leq k1 \leq 1$, and that $x_X = k_1 \cdot x_B$, and $y_X - y_P = k_1 \cdot (y_B - y_P)$, and therefore $\mathcal{B}_t(P, X, B)$ holds. Similarly, we can define a real number k_2 by $\frac{k_4 \cdot (1-k_3)}{k_4 + k_3 - k_3 \cdot k_4}$ and show that $0 \leq k_2 \leq 1$ and that following holds: $x_X - x_Q = -k_2 \cdot x_Q$ and $y_X - y_Q = -k_2 \cdot y_Q$ and thus $\mathcal{B}_t(Q, X, A)$ holds. From these two conclusion we have determined point X.

Lower dimension axiom. The next axiom states that there are 3 non-collinear points. Hence any model of these axioms must have dimension greater than 1. It trivially holds in our Cartesian model (e.g., $(0, 0)$, $(0, 1)$, and $(1, 0)$ are non-collinear.

$\exists\ A\ B\ C.\ \neg\ \mathcal{C}_t(A, B, C)$

Axiom (Schema) of Continuity. Tarski's continuity axiom is essentially the Dedekind cut construction. Intuitively, if all points of a set of points are on one side of all points of the other set of points, then there is a point between the two sets. The original Tarski's are defined within the framework of First Order Logic and sets are not explicitly recognized in Tarski's formalization. Instead of speaking about sets of points, Tarski uses first order predicates ϕ and ψ.

$$(\exists a.\ \forall x.\ \forall y.\ \phi\ x \wedge \psi\ y \longrightarrow \mathcal{B}_t(a, x, y)) \longrightarrow (\exists b.\ \forall x.\ \forall y.\ \phi\ x \wedge \psi\ y \longrightarrow \mathcal{B}_t(x, b, y))$$

However, the formulation of this lemma within the Higher Order Logic framework of Isabelle/HOL does not restrict predicate f and g to be FOL predicates. Therefore, from a strict viewpoint, our formalization of Tarski's axioms within Isabelle/HOL gives a different geometry then Tarski's original axiomatic system.

```
lemma
  assumes "∃a. ∀x. ∀y. φ x ∧ ψ y ⟶ B_T^ag a x y"
  shows "∃b. ∀x. ∀y. φ x ∧ ψ y ⟶ B_T^ag x b y"
```

Still, it turns out that it is possible to show that the Cartesian plane also satisfies the stronger variant of the axiom (without FOL restrictions on predicates f and g). If one of the sets is empty, the statement trivially holds. If the sets have a point in common, that point is the point sought. In other cases, isometry transformations are applied so that all points from both sets lie on the positive part of the y-axis. Then, the statement reduces to proving

```
lemma
  assumes
  "P = {x. x ≥ 0 ∧ φ(0, x)}" "Q = {y. y ≥ 0 ∧ ψ(0, y)}"
  "¬(∃b. b ∈ P ∧ b ∈ Q)" "∃x₀. x₀ ∈ P" "∃y₀. y₀ ∈ Q"
  "∀x ∈ P. ∀y ∈ Q. B_T^ag (0,0) (0,x) (0,y)"
  shows
  "∃b. ∀x ∈ P. ∀y ∈ Q. B_T^ag (0,x) (0,b) (0,y)"
```

Proving this requires using non-trivial properties of reals, i.e., their completeness. Completeness of reals in Isabelle/HOL is formalized in the following theorem (the supremum, i.e., the least upper bound property):

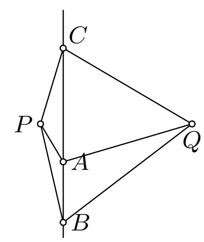
$$(\exists x.\ x \in P) \wedge (\exists y.\ \forall x \in P.\ x < y) \longrightarrow \exists S.\ (\forall y.\ (\exists x \in P.\ y < x) \leftrightarrow y < S)$$

P satisfies the supremum property. Indeed, since, by an assumption, P and Q do not share a common element, from the assumptions it holds that $\forall x \in P.\ \forall y \in Q.\ x < y$, so any element of Q is an upper bound for P. By assumptions, P and Q are non-empty, so there is an element b such that $\forall x \in P.\ x \leq b$ and $\forall y \in Q.\ b \leq y$, so the theorem holds.

5.3 Axioms of Congruence and Betweenness.

Upper dimension axiom. Three points equidistant from two distinct points form a line. Hence any model of these axioms must have dimension less than 3.

$$AP \cong_t AQ \wedge BP \cong_t BQ \wedge CP \cong_t CQ \wedge P \neq Q \longrightarrow \mathcal{C}_t(A, B, C)$$



Segment construction axiom.

$$\exists E.\ \mathcal{B}_t(A, B, E) \wedge BE \cong_t CD$$

The proof that our Cartesian plane models this axiom is simple and starts by transforming the points so that A becomes the origin and that B lies on the positive part of the y-axis. Then $A = (0, 0)$ and $B = (0, b)$, $b \geq 0$. Let $d = \sqrt{d^{2,ag}\ C\ D}$. Then $E = (0, b + d)$.

Five segment axiom.

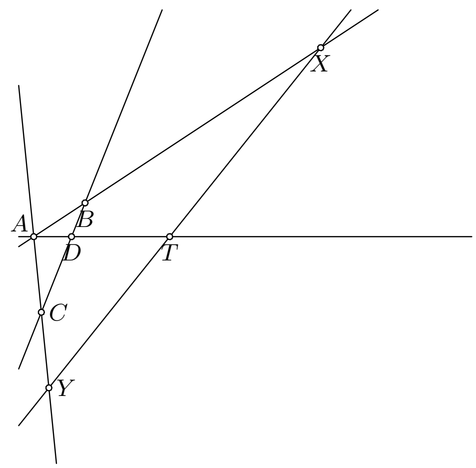
$$AB \cong_t A'B' \wedge BC \cong_t B'C' \wedge AD \cong_t A'D' \wedge BD \cong_t B'D' \wedge$$
$$\mathcal{B}_t(A, B, C) \wedge \mathcal{B}_t(A', B', C') \wedge A \neq B \longrightarrow CD \cong_t C'D'$$

Proving that our model satisfies this axiom was rather straightforward, but it required complex calculations. To simplify the proofs, points A, B and C were transformed to the positive part of the y-axis. Since calculations involved square roots, we did not manage to use much automatization and many small steps needed to be spelled out manually.

The axiom of Euclid.

$$\mathcal{B}_t(A, D, T) \wedge \mathcal{B}_t(B, D, C) \wedge A \neq D \longrightarrow$$
$$(\exists XY.\ (\mathcal{B}_t(A, B, X) \wedge \mathcal{B}_t(A, C, Y) \wedge \mathcal{B}_t(X, T, Y)))$$

The corresponding picture when all points are distinct is:



6 Hilbert's geometry

Our goal in this section is to prove that our definitions of the Cartesian plane satisfy the axioms of Hilbert's geometry. Hilbert's plane geometry considers points, lines, betweenness (denoted by $\mathcal{B}_h(A, B, C)$) and congruence (denoted by $AB \cong_h C$) as basic objects.

In Hilbert original axiomatization [9] some assumptions are implied from the context. For example, if it is said that there exists two points, it always means two distinct points. If this is not assumed then some statements does not hold, e.g. betweenness does not hold if the points are equal. Having this is mind, it could be said that formalizations increase the degree of rigor of Hilbert axioms.

6.1 Axioms of Incidence

$A \neq B \longrightarrow \exists l.\ A \in_h l \wedge B \in_h l$
$A \neq B \longrightarrow \exists! l.\ A \in_h l \wedge B \in_h l$

The final axioms of this groups is formalized within two separate statements.

$\exists AB.\ A \neq B \ \wedge \ A \in_h l \ \wedge B \in_h l$
$\exists ABC.\ \neg \mathcal{C}_h(A, B, C)$

The collinearity relation $\mathcal{C}_h$ (used in the previous statement) is defined as $\mathcal{C}_h(A, B, C) \longleftrightarrow \exists l.\ A \in_h l \ \wedge \ B \in_h l \ \wedge \ C \in_h l$.

Of course, we want to show that our Cartesian plane definition satisfies these axioms. For example, this means that we need to show that

$A \neq B \longrightarrow \exists l.\ A \in_H^{ag} l \wedge B \in_H^{ag} l.$

Proofs of all these lemmas are trivial and mostly done by unfolding the definitions and then using automation (using the Gröbner bases methods).

6.2 Axioms of Order

lemma
  assumes $\mathcal{B}_h(A, B, C)$
  shows $A \neq B \ \wedge \ A \neq C \ \wedge \ B \neq C \ \wedge \ (\mathcal{C}_h(A, B, C)) \ \wedge \ \mathcal{B}_h(C, B, A)$

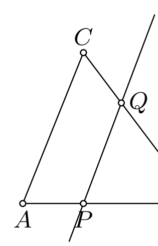
lemma
  assumes $A \neq C$
  shows $\exists B.\ \mathcal{B}_h(A, C, B)$

lemma
  assumes $A \in_h l \ \ B \in_h l \ \ C \in_h l$
  $A \neq B \ \ B \neq C \ \ A \neq C$
  shows $\mathcal{B}_h(A, B, C) \ \wedge \ \neg(\mathcal{B}_h(B, C, A)) \ \wedge \ \neg(\mathcal{B}_h(C, A, B)) \vee$
  $\neg(\mathcal{B}_h(A, B, C)) \ \wedge \ \mathcal{B}_h(B, C, A) \ \wedge \ \neg(\mathcal{B}_h(C, A, B)) \vee$
  $\neg(\mathcal{B}_h(A, B, C)) \ \wedge \ \neg(\mathcal{B}_h(B, C, A)) \ \wedge \ \mathcal{B}_h(C, A, B)$

The proof that the relations $\cong^{ag}$, $\in_H^{ag}$, and $\mathcal{B}_H^{ag}$ satisfy these axioms are simple and again have been done mainly by unfolding the definitions and using automation.

Axiom of Pasch.

lemma
  assumes $A \neq B \ \ B \neq C \ \ C \neq A$
  $\mathcal{B}_h(A, P, B) \ \ P \in_h l \ \ \neg C \in_h l$
  $\neg A \in_h l \ \ \neg B \in_h lh$
  shows $\exists Q.\ (\mathcal{B}_h(A, Q, C) \ \wedge \ Q \in_h l) \ \vee \ (\mathcal{B}_h(B, Q, C) \ \wedge \ Q \in_h l)$



In the original Pash axiom there is one more assumption – points A, B and C are not colinear, so the axiom is formulated only for the central, non-degenerate case. However, in our model the statement holds trivially if they are, so we have shown that our model satisfies both the central and the degenerate case of collinear points. Note that, due to the properties of the Hilbert's between relation, the assumptions about the distinctness of points cannot be omitted.

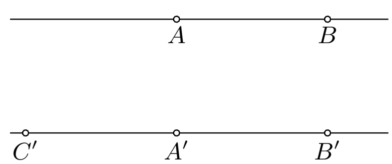
The proof uses the standard technique. First, isometric transformations are used to translate points to the y-axis, so that $A = (0, 0)$, $B = (x_B, 0)$ and $P = (x_P, 0)$. Let $C = (x_C, y_C)$ and $Rep\_line\_coeffs\ l = (l_A, l_B, l_C)$. We distinguish two major cases, depending in which of the given segments requested point lies.

Using the property $\mathcal{B}_h(A,P,B)$ it is shown that $l_A \cdot y_B \neq 0$ and then, two coefficient $k_1 = \frac{-l_C}{l_A \cdot y_B}$ and $k_2 = \frac{l_A \cdot y_B + l_C}{l_A \cdot y_B}$ are defined. Next, it is shown that it holds $0 < k_1 < 1$ or $0 < k_2 < 1$. Using $0 < k_1 < 1$, the point $Q = (x_Q, y_Q)$ is determined by $x_Q = k_1 \cdot x_C$ and $y_Q = k_1 \cdot y_C$, thus $\mathcal{B}_h(A,Q,C)$ holds. In the other case, when the second property holds, the point $Q = (x_q, y_q)$ is determined by $x_Q = k_2 \cdot (x_C - x_B) + x_B$ and $y_Q = k_2 \cdot y_C$, thus $\mathcal{B}_t(B,Q,C)$ holds.

6.3 Axioms of Congruence

The first axiom gives the possibility of constructing congruent segments on given lines. In Hilbert's Grundlagen [9] it is formulated as follows: *If A, B are two points on a line a, and A' is a point on the same or another line a' then it is always possible to find a point B' on a given side of the line a' through A' such that the segment AB is congruent to the segment $A'B'$.* However, in our formalization part *on a given side* is changed and two points are obtained (however, that is implicitly stated in the original axiom).

```
lemma
  assumes "A ≠ B" "A ∈h l" "B ∈h l" "A' ∈h l'"
  shows   "∃B' C'. B' ∈h l' ∧ C' ∈h l' ∧ Bh(C', A', B') ∧
           AB ≅h A'B' ∧ AB ≅h A'C'"
```



The proof that this axiom holds in our Cartesian model, starts with isometric transformation so that A' becomes $(0,0)$ and l' becomes the x-axes. Then, it is rather simple to find the two points on the x-axis by determining their coordinates using condition that $d^{2,ag}$ between them and the point A' is same as the $d^{2,ag}\ A\ B$.

The following two axioms are proved straightforward by unfolding the corresponding definitions, and automatically performing algebraic calculations and Gröbner basis method.

```
lemma
  assumes AB ≅h A'B'  AB ≅h A''B''
  shows A'B' ≅h A''B''
```

```
lemma
  assumes "Bh(A, B, C)" "Bh(A', B', C')" "AB ≅h A'B'" "BC ≅h B'C'"
  shows   "AC ≅h A'C'"
```

Next three axioms in the Hilbert's axiomatization are concerning the notion of angles, and we have not yet considered angles in our formalization.

6.4 Axiom of Parallels

```
lemma
  assumes "¬P ∈_h l"
  shows "∃! l'. P ∈_h l'  ∧  ¬(∃ P_1. P_1 ∈_h l  ∧  P_1 ∈_h l')"
```

The proof of this axiom consists of two parts. First, it is shown that such line exists and second, that it is unique. Showing the existence is done by finding coefficients of the line sought. Let $P = (x_P, y_P)$ and $Rep\_line\_coeffs l = (l_A, l_B, l_C)$. Then coefficients of the requested line are $(l_A, l_B, -l_A \cdot x_P - l_B \cdot y_P)$. In the second part, the proof starts from the assumption that there exists two lines that satisfy the condition $P \in_h l' \ \wedge \ \neg(\exists\ P_1.\ P_1 \in_h l \ \wedge \ P_1 \in_h l')$. In the proof it is shown that their coefficients are proportional and thus the lines are equal.

6.5 Axioms of Continuity

Axiom of Archimedes. Let A_1 be any point upon a straight line between the arbitrarily chosen points A and B. Choose the points $A_2, A_3, A_4, \ldots$ so that A_1 lies between A and A_2, A_2 between A_1 and A_3, A_3 between A_2 and A_4 etc. Moreover, let the segments $AA_1, A_1A_2, A_2A_3, A_3A_4, \ldots$ be equal to one another. Then, among this series of points, there always exists a point A_n such that B lies between A and A_n.

It is rather difficult to represent series of points in a manner as stated in the axiom and our solution was to use list. First, we define a list such that each four consecutive points are congruent and betweenness relation holds for each three consecutive points.

```
definition
  congruentl l → length l ≥ 3 ∧
        ∀i. 0 ≤ i  ∧  i + 2 < length l →
        (l ! i)(l ! (i + 1)) ≅_h (l ! (i + 1))(l ! (i + 2))  ∧
        B_h((l ! i), (l ! (i + 1)), (l ! (i + 2)))
```

Having this, the axiom can be bit transformed, but still with the same meaning, and it states that there exists a list of points with properties mentioned above such that for at least one point A' of the list, $\mathcal{B}_t(A, B, A')$ holds. In Isabelle/HOL this is formalized as:

```
lemma
  assumes B_h(A, A_1, B)
  shows   ∃l. congruentl (A # A1 # l)  ∧  (∃i. B_h(A, B, (l ! i)))
```

The main idea of the proof is in the statements $d^{2,ag}\ A\ A' > d^{2,ag}\ A\ B$ and $d^{2,ag}\ A\ A' = t \cdot d^{2,ag}\ A\ A_1$. So, in the first part of the proof we find such t that $t \cdot d^{2,ag}\ A\ A_1 > d^{2,ag}\ A\ B$ holds. This is achieved by applying Archimedes' rule for real numbers. Next, it is proved that there exists a list l such that `congruentl` l holds, that it is longer then t, and that it's first two elements are A and A_1.

This is done by induction on the parameter t. The basis of induction, when $t = 0$ trivially holds. In the induction step, the list is extended by one point such that it is congruent with the last three elements of the list and that between relation holds for the last two elements and added point. Using these conditions, coordinates of the new point are easily determined by algebraic calculations. Once constructed, the list satisfies the conditions of the axiom, what is easily showed in the final steps of the proof. The proof uses some additional lemmas which are mostly used to describe properties of the list that satisfies condition `congruentl` l.

7 Related work

There are a number of formalizations of fragments of various geometries within proof assistants.

Formalizing Tarski geometry using Coq proof assistant was done by Narboux [16]. Many geometric properties are derived, different versions of Pasch axiom, betweenness and congruence properties.The paper is concluded with the proof of existence of midpoint of a segment.

Another formalization using Coq was done for projective plane geometry by Magaud, Narboux and Schreck [12, 13]. Some basic properties are derived, as well as the principle of duality for projective geometry. Finely the consistency of the axioms are proved in three models, both finite and infinite. In the end authors discuss the degenerate cases and choose ranks and flats to deal with them.

The first attempt to formalize first three groups of Hilbert's axioms and its consequences in Isabelle/Isar proof assistant was done by Meikele and Fleuriot [15]. The authors argue the common believed assumption that Hilbert's proofs are less intuitive and more rigorous. Important conclusion is that Hilbert uses many assumptions that in formalization checked by a computer could not be made and therefore had to be formally justified.

Guilhot connects Dynamic Geometry Software (DGS) and formal theorem proving using Coq proof assistant in order to ease studying the Euclidean geometry for high school students [6]. Pham, Bertot and Narboux suggest a few improvements [18]. The first is to eliminate redundant axioms using a vector approach. They introduced four axioms to describe vectors and tree more to define Euclidean plane, and they introduced definitions to describe geometric concepts. Using this, geometric properties are easily proved. The second improvement is use of area method for automated theorem proving. In order to formally justify usage of the area method, Cartesian plane is constructed using geometric properties previously proved.

Avigad describes the axiomatization of Euclidean geometry [1]. Authors start from the claim that Euclidean geometry describes more naturally geometry statements than some axiomatizations of geometry done recently and it's diagrammatic approach is not so full of weaknesses as some might think. In order to prove this, the system E is introduced in which basic objects such as point, line, circle are described as literals and axioms are used to describe diagram properties from which conclusions could be made. The authors also illustrate the logical framework in which proofs can be constructed. In the work are presented some proofs of geometric properties as well as equivalence between Tarski's system for ruler-and-compass and E. The degenerate cases are avoided by making assumptions and thus only proving general case.

In [20] is proposed the minimal set of Hilbert axioms and set theory is used to model it. The main properties and theorems are carried out within this model.

In many of these formalizations the authors omitted discussion about degenerate cases. Although, usually the general case expresses important geometry property, observing degenerate cases usually leads to conclusion about some basic properties such as transitivity or symmetry, and thus makes them equally important.

Beside formalization of geometries many authors tried to formalize automated proving in geometry.

Grgoire, Pottier and Théry combine a modified version of Buchbergers algorithm and some reflexive techniques to get an effective procedure that automatically produces formal proofs of theorems in geometry [5].

Génevaux, Narboux and Schreck formalize Wu's simple method in Coq [4]. Their approach is based on verification of certificates generated by an implementation in Ocaml of a simple version of Wu's method.

Fuchs and Théry formalize Grassmann-Cayley algebra in Coq proof assistant [3]. The second part, more interesting in the context of this paper, presents application of the algebra on the geometry of incidence. Points, lines and there relationships are defined in form of algebra operations. Using this, theorems of Pappus and Desargues are interactively proved in Coq. Finally the authors describe the automatisation in Coq of theorem proving in geometry using this algebra. The drawback of this work is that only those statements where goal is to prove that some points are collinear can automatically be proved and that only non-degenerate cases are considered.

Pottier presents programs for calculating Grobner basis, F4, GB and gbcoq and compares them [19]. A solution with certificates is proposed and this shortens the time for computation such that gbcoq, although made in Coq, becomes competitive with two others. Application of Gröbner basis on algebra, geometry and arithmetic are represented through three examples.

8 Conclusions and Further Work

In this paper, we have developed a formalization of Cartesian plane geometry within Isabelle/HOL. Several different definitions of the Cartesian plane were given, but it was shown that they are all equivalent. The definitions were taken from the standard textbooks. However, to express them in a formal setting of a proof assistant, much more rigour was necessary. For example, when expressing lines by equations, some textbooks mention that equations represent the line if their coefficients are "proportional", while some other fail even to mention this.

The texts usually do not mention constructions like equivalence relations and equivalence classes that underlie our formal definitions.

We have formally shown that the Cartesian plane satisfies all Tarski's axioms and most of the Hilbert's axioms (including the continuity axiom). Proving that our Cartesian plane model satisfies all the axioms of the Hilbert's system is left for further work (as we found the formulation of the completeness axiom problematic).

Our experience shows that proving that our model satisfies Hilbert's axioms is easier then showing that it satisfies Tarski's axioms. This is mostly due to the definition of the betweenness relation. Namely, Tarski's axiom allows points connected by the betweenness relation to be equal. This gives rise to many degenerate cases that need to be considered separately, what complicates reasoning and proofs.

The fact that analytic geometry models geometric axioms is usually taken for granted, as a rather simple fact. However, our experience shows that, although conceptually simple, the proof of this fact requires complex computations and is very demanding for formalization. It turned out that the most significant technique used to simplify the proof was "without loss of generality reasoning" and using isometry transformations. For example, we have tried to prove the central case of the Pasch's axiom, without applying isometry transformations first. Although it should be possible do a proof like that, the arising calculations were so difficult that we did not manage to finish such a proof. After applying isometry transformations, calculations remained non-trivial, but still, we managed to finish this proof (however, many manual interventions had to be used because even powerful tactics relying on the Gröbner bases did not manage to do all the algebraic simplifications automatically). From this experiment on Pasch's axiom, we learned the significance of isometry transformations and we did not even try to prove other lemmas directly.

Our formalization of the analytic geometry relies on the axioms of real numbers and properties of reals are used throughout our proofs. Many properties would hold for any numeric field (and Gröbner bases tactics used in our proofs would also work in that case). However, for showing the continuity axioms, we used the supremum property, not holding in an arbitrary field. In our further work, we would like to build analytic geometries without using the axioms of real numbers, i.e., define analytic geometries within Tarski's or Hilbert's axiomatic system. Together with the current work, this would help analyzing some model theoretic properties of geometries. For example, we want to show the categoricity of both Tarski's and Hilbert's axiomatic system (and prove that all models are isomorphic and equivalent to the Cartesian plane).

Our present and further work also includes formalizing analytic models of non-Euclidean geometries. For example, we have given formal definitions of the Poincaré disk (were points are points in the unit disk and lines are circle segments perpendicular to the unit circle) using the Complex numbers available in Isabelle/HOL and currently we are showing that these definitions satisfy all axioms except the parallelness axiom.

Finally, we want to connect our formal developments to the implementation of algebraic methods for automated deduction in geometry, making formally verified yet efficient theorem provers for geometry.

References

1. Jeremy Avigad, Edward Dean, and John Mumma. A formal system for Euclid's *Elements*. *Review of Symbolic Logic*, 2(4):700–768, 2009.
2. Bruno Buchberger. Bruno buchberger's phd thesis 1965: An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal. *J. Symb. Comput.*, 41(3-4):475–511, 2006.
3. Laurent Fuchs and Laurent Théry. A formalization of grassmann-cayley algebra in coq and its application to theorem proving in projective geometry. In *Proceedings of the 8th international conference on Automated Deduction in Geometry*, ADG'10, pages 51–67, Berlin, Heidelberg, 2011. Springer-Verlag.
4. Jean-David Génevaux, Julien Narboux, and Pascal Schreck. Formalization of wu's simple method in coq. In Jean-Pierre Jouannaud and Zhong Shao, editors, *CPP*, volume 7086 of *Lecture Notes in Computer Science*, pages 71–86. Springer, 2011.
5. Benjamin Grégoire, Loïc Pottier, and Laurent Théry. Proof certificates for algebra and their application to automatic geometry theorem proving. In *Automated Deduction in Geometry - 7th International Workshop, ADG 2008, Shanghai, China, September 22-24, 2008. Revised Papers*, volume 6301 of *Lecture Notes in Computer Science*, pages 42–59. Springer, 2011.
6. F. Guilhot. Formalisation en coq et visualisation dun cours de géométrie pour le lycé. *TSI 24, 11131138*, 2005.
7. Jean-David Gnevaux, Julien Narboux, and Pascal Schreck. Formalization of wu's simple method in coq. In Jean-Pierre Jouannaud and Zhong Shao, editors, *CPP 2011*, volume 7086 of *LNCS*. Springer-Verlag, Dec 2011.
8. John Harrison. Without loss of generality. In Stefan Berghofer, Tobias Nipkow, Christian Urban, and Makarius Wenzel, editors, *TPHOLs*, volume 5674 of *Lecture Notes in Computer Science*, pages 43–59. Springer, 2009.
9. David Hilbert. *Grundlagen der Geometrie.* Leipzig, B.G. Teubner, 1903.
10. Cezary Kaliszyk and Christian Urban. Quotients revisited for isabelle/hol. In William C. Chu, W. Eric Wong, Mathew J. Palakal, and Chih-Cheng Hung, editors, *SAC*, pages 1639–1644. ACM, 2011.
11. Deepak Kapur. Using grb̈ner bases to reason about geometry problems. *Journal of Symbolic Computation*, 2(4), 1986.
12. Nicolas Magaud, Julien Narboux, and Pascal Schreck. Formalizing projective plane geometry in coq. In Thomas Sturm and Christoph Zengler, editors, *Automated Deduction in Geometry*, volume 6301 of *Lecture Notes in Computer Science*, pages 141–162. Springer, 2008.
13. Nicolas Magaud, Julien Narboux, and Pascal Schreck. A case study in formalizing projective geometry in coq: Desargues theorem. *Comput. Geom.*, 45(8):406–424, 2012.
14. Filip Marić, Ivan Petrović, Danijela Petrović, and Predrag Janičić. Formalization and implementation of algebraic methods in geometry. In Pedro Quaresma and Ralph-Johan Back, editors, Proceedings First Workshop on *CTP Components for Educational Software,* Wrocław, Poland, 31th July 2011, volume 79 of *Electronic Proceedings in Theoretical Computer Science*, pages 63–81. Open Publishing Association, 2012.

15. Laura I. Meikle and Jacques D. Fleuriot. Formalizing hilbert's grundlagen in isabelle/isar. In David A. Basin and Burkhart Wolff, editors, *TPHOLs*, volume 2758 of *Lecture Notes in Computer Science*, pages 319–334. Springer, 2003.
16. Julien Narboux. Mechanical theorem proving in tarski's geometry. In Francisco Botana and Tomás Recio, editors, *Automated Deduction in Geometry*, volume 4869 of *Lecture Notes in Computer Science*, pages 139–156. Springer, 2006.
17. Tobias Nipkow, Lawrence C. Paulson, and Markus Wenzel. *Isabelle/HOL - A Proof Assistant for Higher-Order Logic*, volume 2283 of *Lecture Notes in Computer Science*. Springer, 2002.
18. Tuan-Minh Pham, Yves Bertot, and Julien Narboux. A coq-based library for interactive and automated theorem proving in plane geometry. In Beniamino Murgante, Osvaldo Gervasi, Andrés Iglesias, David Taniar, and Bernady O. Apduhan, editors, *ICCSA (4)*, volume 6785 of *Lecture Notes in Computer Science*, pages 368–383. Springer, 2011.
19. Loïc Pottier. Connecting gröbner bases programs with coq to do proofs in algebra, geometry and arithmetics. *CoRR*, abs/1007.3615, 2010.
20. William Richter. A minimal version of hilbert's axions for plane geometry.
21. W. Schwabhuser, W. Szmielew, and A. Tarski. *Metamathematische Methoden in der Geometrie*. Springer-Verlag, 1983.
22. Wen-Tsün Wu. On the decision problem and the mechanization of theorem proving in elementary geometry. *Scientia Sinica*, 21, 1978.