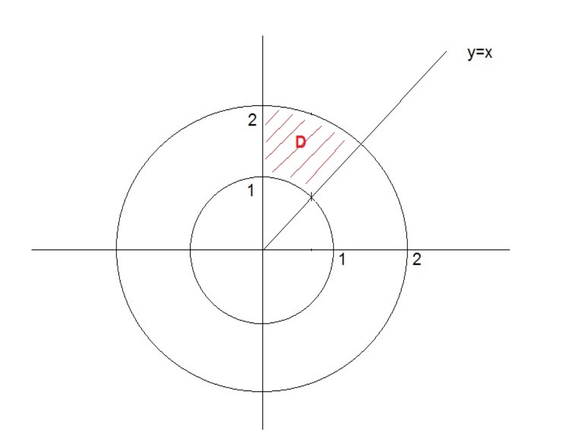
Integracija - smene promenljivih

Polarne koordinate

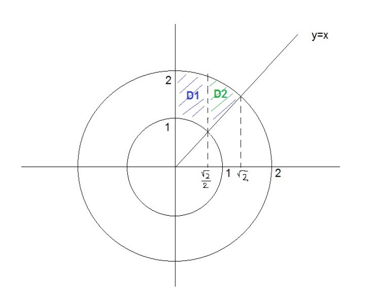
NAPOMENA: Podsetiti se Jakobijana!

Primer

Neka je D oblast ograničena kružnicama sa centrom u $(0,0)$, *poluprečnika* $r_1 = 1$ *i* $r_2 = 2$ *i pravama* $y = x$ *i* $x = 0$.



I način:



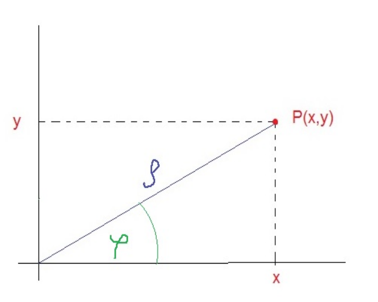
$D = D1 \cup D2$

$D1 = \{(x,y) | x \in [0, \frac{\sqrt{2}}{2}], y \in [\sqrt{1-x^2}, \sqrt{4-x^2}]\}$

$D2 = \{(x,y) | x \in [\frac{\sqrt{2}}{2}, \sqrt{2}], y \in [x, \sqrt{4-x^2}]\}$

$$\int\int_{D} f(x,y)dxdy = \int\int_{D1} f(x,y)dxdy + \int\int_{D2} f(x,y)dxdy$$

II način: koristeći polarne koordinate



$P(x, y)$ - tačka

ρ - rastojanje tačke P od koordinatnog početka, $\rho \geq 0$

φ - ugao između x-ose i vektora određenog tačkom P, $\varphi \in [0, 2\pi]$

$$\begin{cases} \sin(\varphi) = \frac{\textit{naspramna}}{\textit{hipotenuza}} = \frac{y}{\rho} \Rightarrow y = \rho \sin(\varphi) \\ \cos(\varphi) = \frac{\textit{nalegla}}{\textit{hipotenuza}} = \frac{x}{\rho} \Rightarrow x = \rho \cos(\varphi) \end{cases}$$

Polarne koordinate:

$x = \rho \cos(\varphi)$
$y = \rho \sin(\varphi)$
$\rho \geq 0,\ \varphi \in [0, 2\pi]$

$$\begin{aligned} x^2 + y^2 &= \rho^2 \cos^2(\varphi) + \rho^2 \sin^2(\varphi) \\ &= \rho^2(\cos^2(\varphi) + \sin^2(\varphi)) \\ &= \rho^2 \\ &\Rightarrow \rho = \sqrt{(x^2 + y^2)} \end{aligned}$$

$$\begin{aligned} \frac{y}{x} &= \frac{\rho \sin(\varphi)}{\rho \cos(\varphi)} \\ &= tg(\varphi) \\ &\Rightarrow \varphi = arctg(\frac{y}{x}) \end{aligned}$$

Nastavak primera:

- Sve tacke koje se nalaze između dve kružnice su na udaljenosti od koordinanog početka za $1 \leq \rho \leq 2$.
- Ugao između prave $y = x$ i x-ose je $\frac{\pi}{4}$, a ugao između prave $x = 0$ i x-ose je $\frac{\pi}{2}$. Za sve tačke između ove dve prave važi da je $\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}$.

$$\Rightarrow D^* = \{(\rho, \varphi) : \rho \in [1, 2], \varphi \in [\tfrac{\pi}{4}, \tfrac{\pi}{2}]\}$$

Jednostavnosti radi, neka je $f(x, y) = 1$:

$$\int\int_D f(x, y)dxdy = \int\int_D 1 \cdot dxdy = \int\int_{D^*} \rho d\rho d\varphi = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_1^2 \rho d\rho d\varphi$$

Odakle ovo ρ ?

Teorema

Uopštena teorema o smeni promenljivih:

Ako je $x = x(u, v)$, $y = y(u, v)$, $J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$ *i* $\det(J) \neq 0$*, tada:*

$$\iint\limits_{D} f(x, y)dxdy = \iint\limits_{D*} f(x(u, v), y(u, v)) \cdot \det(J)dudv$$

Teorema

Neka je funkcija $f(x, y)$ *neprekidna na* $D \subseteq \mathbb{R}^2$ *koji se pomoću* ***polarnih koordinata*** *može transformisati u* $D^* = \{(\rho, \varphi) : \rho \in [a, b], \varphi \in [\alpha, \beta]\}$*. Tada je:*

$$\iint\limits_{D} f(x, y)dxdy = \iint\limits_{D*} f(x(\rho, \varphi), y(\rho, \varphi)) \cdot \rho d\rho d\varphi$$

Polarne koordinate

Polarne koordinate:
$x = \rho\cos(\varphi)$
$y = \rho\sin(\varphi)$
$\rho \geq 0$, $\varphi \in [0, 2\pi]$,
$|J| = \rho$.

Smenom: $x = \rho\cos(\varphi)$, $y = \rho\sin(\varphi)$ posmatramo preslikavanje čija je Jakobijeva matrica

$$J = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \cos(\varphi) & -\rho\sin(\varphi) \\ \sin(\varphi) & \rho\cos(\varphi) \end{bmatrix}.$$

$$\begin{aligned} |J| = \det(J) &= \cos(\varphi)\cdot\rho\cos(\varphi) + \rho\sin(\varphi)\cdot\sin(\varphi) \\ &= \rho(\cos^2(\varphi) + \sin^2(\varphi)) \\ &= \rho \end{aligned}$$

Eliptičke koordinate

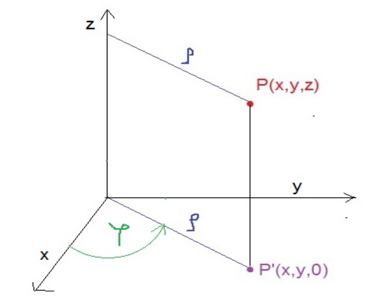
Eliptičke koordinate:
$a, b \in \mathbb{R}$, $\rho \geq 0$, $\varphi \in [0, 2\pi]$
$x = a\rho\cos(\varphi)$
$y = b\rho\sin(\varphi)$
$|J| = ab\rho.$

Imamo da je :

$$J = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} \end{bmatrix} = \begin{bmatrix} a\cos(\varphi) & -a\rho\sin(\varphi) \\ b\sin(\varphi) & b\rho\cos(\varphi) \end{bmatrix}.$$

$$\begin{aligned} |J| = \det(J) &= ab\cos(\varphi)\cdot\rho\cos(\varphi) + ab\rho\sin(\varphi)\cdot\sin(\varphi) \\ &= ab\rho(\cos^2(\varphi) + \sin^2(\varphi)) \\ &= ab\rho. \end{aligned}$$

Cilindrične koordinate



$P(x, y, z)$ - tačka

$P'(x, y, 0)$ - projekcija tačke P na Oxy ravan

ρ - rastojanje tačke P' od koordinatnog početka, $\rho \geq 0$

φ - ugao između x-ose i vektora određenog tačkom P', $\varphi \in [0, 2\pi]$

Za zapis projekcije tačaka cilindra na Oxy ravan koriste se polarne koordinate. Treća promenljiva z ostaje nepromenjena.

$P'(x, y, 0)$
$P'(\rho\cos(\varphi), \rho\sin(\varphi), 0)$

$P(x, y, z)$
$P(\rho\cos(\varphi), \rho\sin(\varphi), z)$

Cilindrične koordinate:

$x = \rho \cos(\varphi)$

$y = \rho \sin(\varphi)$

$z = z$

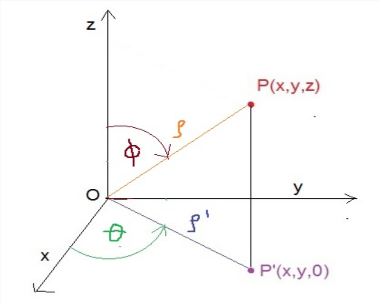
$\rho \geq 0$, $\varphi \in [0, 2\pi], z \in \mathbb{R}$

$|J| = \rho$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos(\varphi) & -\rho \sin(\varphi) & 0 \\ \sin(\varphi) & -\rho \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos(\varphi) & -\rho \sin(\varphi) \\ \sin(\varphi) & -\rho \cos(\varphi) \end{vmatrix} = \rho$$

Sferne koordinate



$P(x, y, z)$ - tačka

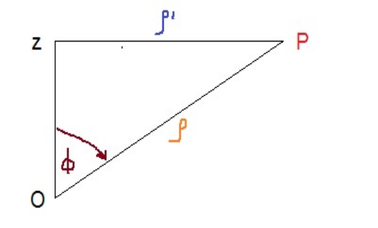
$P'(x, y, 0)$ - projekcija tačke P na Oxy ravan

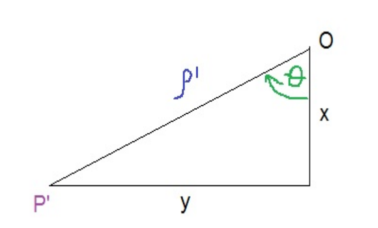
ρ - rastojanje tačke P od koordinatnog početka, $\rho \geq 0$

ρ' - rastojanje tačke P' od koordinatnog početka, $\rho' \geq 0$

θ - ugao između x-ose i OP', $\theta \in [0, 2\pi]$

ϕ - ugao između z-ose i OP, $\phi \in [0, \pi]$





$$\sin(\phi) = \frac{\rho'}{\rho} \Rightarrow \rho' = \rho \sin(\phi)$$
$$\cos(\phi) = \frac{z}{\rho} \Rightarrow z = \rho \cos(\phi)$$

$$\sin(\theta) = \frac{y}{\rho'} \Rightarrow y = \rho' \sin(\theta)$$
$$\cos(\theta) = \frac{x}{\rho'} \Rightarrow x = \rho' \cos(\theta)$$

Hiperboličke koordinate:

$x = a\rho \sin(\phi) \cos(\theta)$

$y = b\rho \sin(\phi) \sin(\theta)$

$z = c\rho \cos(\phi)$

$a, b, c \in \mathbb{R}$, $\rho \geq 0$, $\theta \in [0, 2\pi]$, $\phi \in [0, \pi]$

$|J| = abc\rho^2 \sin(\phi)$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = abc\rho^2 \sin(\phi)$$

Specijalan slučaj (hiperboličkih kada je $a = b = c = 1$):

Sferne koordinate:

$x = \rho \sin(\phi) \cos(\theta)$
$y = \rho \sin(\phi) \sin(\theta)$
$z = \rho \cos(\phi)$
$\rho \geq 0$, $\theta \in [0, 2\pi]$, $\phi \in [0, \pi]$
$|J| = \rho^2 \sin(\phi)$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \rho^2 \sin(\phi)$$