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COSMIC TIME FOR MULTI-COMPONENT UNIVERSE

D. Branković

Department of Applied Mathematics, School of Electrical Engineering, University of Belgrade
Bulevar kralja Aleksandra 73, 11000 Belgrade, Serbia

E–mail: *danijela@etf.bg.ac.rs*

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SUMMARY: We determined Weierstrass canonical form of cosmic time formula in the cases of four-component and some three-component universe, assuming the ΛCDM model. In all other cases, we discussed analytical solutions for the cosmic time formula.

Key words. Cosmological parameters

1. INTRODUCTION

Our aim in this paper is to discuss all cases that consider the existence of analytical form of cosmic time formula for a multi-component universe obeying the ΛCDM model. In the cases when the analytical solution is not available, we will obtain the Weierstrass canonical form of the appropriate integrals.

Having in mind the physical constraints, we restrict here our attention only to smooth functions, i.e. those having at least the continuous second derivative. The additional reason for this premise is that all cosmological parameters that we study here are the solutions of Friedmann equations (see Friedmann 1924) with the Λ term, a system consisting of the first and second order differential equations:

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}, \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}, \\ \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) &= 0. \end{aligned} \tag{1}$$

The first equation in Eq. (1) refers to the Friedmann equation, the second one is known as the acceleration equation, while the last one is the fluid equation. Parameters appearing in these equations are the scale expansion factor $a(t)$ in the time variable t, the pressure $p(t)$ of the material in the universe, the energy density parameter $\rho(t)$ and the universe's curvature k. Here we consider only the cases $k = 0$ (flat universe) and $k < 0$ (open universe), wherefrom it follows that the curvature density parameter, given with $\Omega_k = -kc^2/a^2H^2$, has nonnegative values. Moreover, all other density parameters, defined as (see Liddle and Lyth 2000):

$$\begin{aligned} \Omega_\Lambda(t) &= \frac{\Lambda c^2}{3H(t)^2}, \\ \Omega(t) &= \frac{8\pi G}{3H(t)^2}\rho(t), \\ \Omega_m(t) &= \frac{8\pi G}{3H(t)^2}\rho_m(t), \\ \Omega_r(t) &= \frac{8\pi G}{3H(t)^2}\rho_r(t), \end{aligned} \tag{2}$$

are also nonnegative (see Zyla et al. 2020), where $\Omega_\Lambda(t)$ denotes the cosmological constant density, $\Omega(t)$ is the mass density, while $\Omega_m(t)$ and $\Omega_r(t)$ denote the rest mass density and radiation density, respectively. Also, they satisfy the identity:

$$\Omega_r(t) + \Omega_m(t) + \Omega_k(t) + \Omega_\Lambda(t) = 1, \tag{3}$$

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which is the another form of the Friedmann equation. Those properties of the density parameters were essential for simplifying the obtained solutions of some integrals in this paper.

Also, some of the density parameters have very small values (see Aghanim et al. 2019), therefore one could consider applying some perturbation methods (for example, see Filobello-Nino et al. 2013) in order to solve approximately elliptic integrals that we study in this paper. Some of the cases that we analyze here are already discussed in literature (see Ryden 2002 and Steiner 2008) but mostly using some approximations.

The evolution of the universe can be roughly divided into four epochs which are determined by one dominant component. The first stage in the universe's evolution was inflation, which was dominated by the Λ-like term that drove the exponential expansion of the universe. After the inflation period, radiation prevailed the universe, until the time of the matter-radiation density equality which occurred at approximately 50 000 years after the Big Bang. At that time, the transition period between the radiation dominated and matter dominated universe began and it lasted until about 400 000 yrs when recombination occurred. Afterwards, the universe was matter-dominated until the time of the matter-dark energy density equality which occurred at approximately 10^9 yrs, when the transition period between the matter dominated and dark energy dominated universe began. Currently, our universe is governed by dark energy, which caused its accelerated expansion which started around $7.7 \cdot 10^9$ yrs.

We note that each case of the elliptic integral we consider in this paper represents some model of the universe, i.e. some stage, or middle stage in the universe's evolution.

2. PARAMETRIZATIONS OF FRIEDMANN EQUATIONS

In this section we consider some parametrizations of Friedmann Eqs. (1). Let t_0 denotes some fixed moment, for example the present time. The value of a parameter $P(t)$ at t_0 we denote by P_0, i.e. $P_0 = P(t_0)$. We assume that $a(t)$ is normalized at t_0, i.e. $a_0 = a(t_0) = 1$. In Mijajlović et al. (2019) the following parametrization is derived:

Theorem. *The first Friedmann equation with non-zero cosmological constant Λ is equivalent to the following equation*:

$$\begin{aligned} &\dot{a}^2 + \frac{8\pi G}{3}\left(\frac{\rho_0}{a^3} - \rho\right) a^2 \\ &= H_0^2 \left(1 + \Omega_0 \left(\tfrac{1}{a} - 1\right) + \Omega_{\Lambda 0}(a^2 - 1)\right). \end{aligned} \tag{4}$$

We remind that the cosmological constant or vacuum density is $\Omega_\Lambda = \Lambda c^2 / 3H^2$. Using the above theorem and the following form of the Friedmann equation:

$$\Omega + \Omega_k + \Omega_\Lambda = 1, \tag{5}$$

we obtain the following parametrizations of the Friedmann equation:

$$\frac{H^2}{H_0^2} = \Omega_{\Lambda 0} + \Omega_{k0} a^{-2} + \Omega_0 a^{-3} + \frac{8\pi G}{3H_0^2}(\rho - \frac{\rho_0}{a^3}), \tag{6}$$

$$\frac{H^2}{H_0^2} = \Omega_{\Lambda 0} + \Omega_{k0} a^{-2} + \Omega_0 \frac{\rho}{\rho_0}. \tag{7}$$

Let ρ_m denote the sum of dark matter and baryonic rest mass density and ρ_r the radiation density. Since $\rho_m = \rho_{m0} a^{-3}$ and $\rho_r = \rho_{r0} a^{-4}$, we have:

$$\rho = \rho_m + \rho_r \quad \text{and} \quad \rho = \rho_{m0} a^{-3} + \rho_{r0} a^{-4}. \tag{8}$$

From Eqs. (6) and (8) we obtain the well known parametrization of the Friedmann equation:

$$\frac{H^2}{H_0^2} = \Omega_{\Lambda 0} + \Omega_{k0} a^{-2} + \Omega_{m0} a^{-3} + \Omega_{r0} a^{-4}. \tag{9}$$

In Mijajlović and Branković (2020) it is proven that the form Eq. (9) of the Friedmann equation is equivalent to the system of the Friedmann Eqs. (1) assuming the identities in Eq. (8) as well as $p = \frac{1}{3} c^2 \rho_r$. Therefore, the identity in Eq. (9) is fundamental for analyzing cosmological parameters. Now let:

$$S(a) = \Omega_{r0} + \Omega_{m0} a + \Omega_{k0} a^2 + \Omega_{\Lambda 0} a^4. \tag{10}$$

By Eq. (9), we have:

$$H_0 \mathrm{d}t = \frac{a}{\sqrt{S(a)}} \mathrm{d}a. \tag{11}$$

Integrating the identity in Eq. (11) over the interval $(0, t_0)$ with respect to t and performing the change of the variable t by $x = a(t)$, as well as having in mind that $a(0) = 0$ and $a(t_0) = 1$, we obtain:

$$\begin{aligned} I = H(t_0) t_0 &= \int_0^1 \frac{x}{\sqrt{S(x)}} \mathrm{d}x \\ &= \int_0^1 \frac{x}{\sqrt{\Omega_{r0} + \Omega_{m0} x + \Omega_{k0} x^2 + \Omega_{\Lambda 0} x^4}} \mathrm{d}x, \end{aligned} \tag{12}$$

when the integral on the righthand side exists.

We note that t_0 may represent any other moment τ. Having that in mind, in Mijajlović and Branković (2020) are introduced new variables x and y by:

$$x = 1/a(\tau) \quad \text{and} \quad y = H_0^2 / H_\tau^2, \tag{13}$$

in order to obtain the following relations between the cosmological parameters:

$$\begin{aligned} &a(\tau) = 1/x, \quad H_\tau = H_0/\sqrt{y}, \\ &\Omega_{\Lambda\tau} = \Omega_{\Lambda 0} y, \quad \Omega_{k\tau} = \Omega_{k0} x^2 y, \\ &\Omega_{m\tau} = \Omega_{m0} x^3 y, \quad \Omega_{r\tau} = \Omega_{r0} x^4 y, \\ &y = \frac{1}{\Omega_{\Lambda 0} + \Omega_{k0} x^2 + \Omega_{m0} x^3 + \Omega_{r0} x^4}. \end{aligned} \tag{14}$$

Suppose that $H(t_0)$ and $\Omega_i(t_0)$ are known. Then, for a given value of x, we can compute by Eq. (14) all parameters at time τ. Furthermore, using Eqs. (9) and (14), it can be obtained (see Mijajlović and Branković 2020):

$$\begin{aligned} I &= H(\tau)\tau \\ &= \int_0^1 \frac{s\,ds}{\sqrt{\Omega_{r\tau}+\Omega_{m\tau}s+\Omega_{k\tau}s^2+\Omega_{\Lambda\tau}s^4}}. \end{aligned} \tag{15}$$

Notice that for $\tau = t_0$ the integral I in Eq. (15) is the same as the integral in Eq. (12). So, from now on, we consider the integral I in Eq. (12).

Since the integral I in Eq. (12) is an elliptic integral, it is not possible in general to find its analytical solution. In the following we will analyze in which cases the analytical solution does exist.

2.1. Four-Component Universe

Here we consider that all Ω_{i0} have positive values, what corresponds to the flat or hyperbolic universe. Since we can not neglect any of the Ω_{i0} parameters, this stage occurred around the time of domination of dark energy over radiation, which was at $\tau \approx 5 \cdot 10^8$ yrs (see Mijajlović and Branković 2020). Note that this case formally suites the matter-dominated epoch since at the evolution timeline, it is approximately in the middle of this period.

We reduce the integral I in Eq. (12) to the Weierstrass form by an appropriate substitution of variables. We follow the methods presented in D'Ambroise and Williams (2011) and D'Ambroise (2009).

First we assume that all roots of the polynomial $S(x) = \Omega_{r0}+\Omega_{m0}x+\Omega_{k0}x^2+\Omega_{\Lambda0}x^4$ are simple. Otherwise, the situation is easier and it does not involve an elliptic integral since, in that case, it is possible to find the integral I in a closed form.
We represent $S(x)$ by its Taylor polynomial of the fourth degree about x_0, where x_0 is an arbitrary simple root of the polynomial $S(x)$. The change of variables $z = 1/(x-x_0)$ transforms the integral I into:

$$\int_{\frac{1}{(1-x_0)}}^{-\frac{1}{x_0}} \frac{(x_0+\frac{1}{z})\mathrm{d}z}{\sqrt{S'(x_0)z^3+\frac{S''(x_0)z^2}{2}+\frac{S'''(x_0)z}{6}+\frac{S^{(4)}(x_0)}{24}}}.$$

Since x_0 is a simple root of the polynomial $S(x)$, it follows that $S'(x_0) \neq 0$. Having that in mind, we substitute $z = (4s - S''(x_0)/6)/S'(x_0)$ in the previous equation, wherefrom we obtain the Weierstrass canonical form of the elliptic integral I:

$$I = \int_{s_1}^{s_2} \frac{x_0 + S'(x_0)/\left(4s - S''(x_0)/6\right)}{\sqrt{4s^3-g_2s-g_3}}\,\mathrm{d}s, \tag{16}$$

where:

$$s_1 = \frac{S'(x_0)}{4(1-x_0)} + \frac{S''(x_0)}{24}, \tag{17}$$

$$s_2 = \frac{-S'(x_0)}{4x_0} + \frac{S''(x_0)}{24}, \tag{18}$$

$$S'(x_0) = \Omega_{m0} + 2\Omega_{k0}x_0 + 4\Omega_{\Lambda0}{x_0}^3, \tag{19}$$

$$S''(x_0) = 2\Omega_{k0} + 12\Omega_{\Lambda0}{x_0}^2. \tag{20}$$

Here:

$$g_2 = \frac{{\Omega_{k0}}^2}{12} + \Omega_{\Lambda0}\Omega_{r0}, \tag{21}$$

$$g_3 = \frac{\Omega_{r0}\Omega_{k0}\Omega_{\Lambda0}}{6} - \frac{{\Omega_{m0}}^2\Omega_{\Lambda0}}{16} - \frac{{\Omega_{k0}}^3}{216}, \tag{22}$$

are the Weierstrass invariants of the polynomial $S(x)$. Since the Weierstrass function $\wp(\omega) = \wp(\omega; g_2, g_3)$, attached to g_2 and g_3, satisfies
${\wp'}^2 = 4\wp^3 - g_2\wp - g_3$ (see Prasolov and Solovyev 1997), by substitution $s = \wp(\omega)$ in Eq. (16), we infer (see D'Ambroise and Williams 2011):

$$I = \int_{\wp^{-1}(s_1)}^{\wp^{-1}(s_2)} \left(x_0 + \frac{S'(x_0)}{4\wp(\omega) - S''(x_0)/6}\right)\mathrm{d}\omega. \tag{23}$$

2.2. Three-Component Universe

Now we consider the cases when only one of four Ω_{i0} is equal to zero. Depending on the obtained integral, we will determine its Weierstrass canonical form, or calculate it in analytical form. In all of the following cases, it is assumed that the universe is dominated by choosing three of four components: radiation, matter, curvature and cosmological constant. Therefore, in this section we have four cases and some of them are discussed in Ryden (2002), without entering into details of solving the appropriate integrals.

Case $\Omega_{r0} = 0$, $\Omega_{m0} > 0$, $\Omega_{k0} > 0$ and $\Omega_{\Lambda0} > 0$. This case corresponds to pressureless and open universe with the cosmological constant. Since the radiation contribution is neglected, this stage occurred around the transition time from the matter to dark energy period, approximately at $\tau \approx 10^{10}$ yrs. The integral I in Eq. (12) has the following form:

$$I = \int_0^1 \frac{x}{\sqrt{\Omega_{m0}x + \Omega_{k0}x^2 + \Omega_{\Lambda0}x^4}}\,\mathrm{d}x. \tag{24}$$

It is obvious that $x = 0$ is a simple root of the polynomial $S(x) = \Omega_{m0}x + \Omega_{k0}x^2 + \Omega_{\Lambda0}x^4$. Since the integral in Eq. (24) is elliptic, following the method presented in the previous section we obtain the Weierstrass canonical form of the elliptic integral I:

$$I = \int_{s_1}^{s_2} \frac{S'(0)/\left(4s - S''(0)/6\right)}{\sqrt{4s^3-g_2s-g_3}}\,\mathrm{d}s, \tag{25}$$

where:

$$s_1 = \frac{S'(0)}{4} + \frac{S''(0)}{24} \quad \text{and} \quad s_2 = +\infty, \tag{26}$$

as well as:

$$S'(0) = \Omega_{m0} \quad \text{and} \quad S''(0) = 2\Omega_{k0}. \tag{27}$$

Here:

$$g_2 = \frac{{\Omega_{k0}}^2}{12}, \tag{28}$$

$$g_3 = -\frac{{\Omega_{m0}}^2\Omega_{\Lambda 0}}{16} - \frac{{\Omega_{k0}}^3}{216}, \tag{29}$$

are the Weierstrass invariants of the polynomial $S(x) = \Omega_{m0}x + \Omega_{k0}x^2 + \Omega_{\Lambda 0}x^4$.
Since the Weierstrass function $\wp(\omega) = \wp(\omega; g_2, g_3)$, attached to g_2 and g_3, satisfies
${\wp'}^2 = 4\wp^3 - g_2\wp - g_3$, by substitution $s = \wp(\omega)$ and Eq. (27) in Eq. (25) we infer:

$$I = \int_{\wp^{-1}(s_1)}^{\wp^{-1}(s_2)} \left(\frac{\Omega_{m0}}{4\wp(\omega) - \Omega_{k0}/3} \right) \mathrm{d}\omega. \tag{30}$$

Case $\Omega_{m0} = 0$, $\Omega_{r0} > 0$, $\Omega_{k0} > 0$ and $\Omega_{\Lambda 0} > 0$. This case corresponds to the radiation dominated and open universe with cosmological constant. Since the matter contribution is neglected, this stage may occurred in the early universe, since in some literature (see Poulin et al. 2019, Niedermann and Sloth 2019), it is proposed that the early dark energy may altered the expansion of the universe at that time. However, this theory is not concordant with the ΛCDM model, therefore this case is not possible in the frame of the standard cosmological model. Nevertheless, we will consider it for completeness of our paper. The integral I in Eq. (12) has the following form:

$$I = \int_0^1 \frac{x}{\sqrt{\Omega_{r0} + \Omega_{k0}x^2 + \Omega_{\Lambda 0}x^4}} \mathrm{d}x. \tag{31}$$

The integral in Eq. (31) is not elliptic, therefore we can find its analytical form, which is:

$$I = \frac{-\ln(\Omega_{k0} + 2\sqrt{\Omega_{r0}}\sqrt{\Omega_{\Lambda 0}})}{2\sqrt{\Omega_{\Lambda 0}}} + \frac{\ln(\Omega_{k0} + 2\Omega_{\Lambda 0} + 2\sqrt{\Omega_{\Lambda 0}}\sqrt{\Omega_{r0} + \Omega_{k0} + \Omega_{\Lambda 0}})}{2\sqrt{\Omega_{\Lambda 0}}}. \tag{32}$$

By Eq. (5) we have $\Omega_{r0} + \Omega_{k0} + \Omega_{\Lambda 0} = 1$. Substituting in Eq. (32), we have:

$$I = \frac{1}{2\sqrt{\Omega_{\Lambda 0}}} \ln\left(\frac{\Omega_{k0} + 2\Omega_{\Lambda 0} + 2\sqrt{\Omega_{\Lambda 0}}}{\Omega_{k0} + 2\sqrt{\Omega_{\Lambda 0}\Omega_{r0}}} \right). \tag{33}$$

Furthermore, taking for example $\Omega_{k0} = 1 - \Omega_{r0} - \Omega_{\Lambda 0}$ in Eq. (33), we finally obtain:

$$I = \frac{1}{2\sqrt{\Omega_{\Lambda 0}}} \ln\left(\frac{1 + \sqrt{\Omega_{\Lambda 0}} + \sqrt{\Omega_{r0}}}{1 - \sqrt{\Omega_{\Lambda 0}} + \sqrt{\Omega_{r0}}} \right). \tag{34}$$

Case $\Omega_{k0} = 0$, $\Omega_{r0} > 0$, $\Omega_{m0} > 0$ and $\Omega_{\Lambda 0} > 0$. For this model of the universe, in Ryden (2002), the term Benchmark model is used. This case corresponds to the flat universe with a mixture of matter, radiation and with the cosmological constant, which, according to the current probes, most likely corresponds to our universe. Like in the previous section, this stage occurred in the matter-dominated epoch, around the time of domination of dark energy over radiation, which was at $\tau \approx 5 \cdot 10^8$ yrs. The integral I in Eq. (12) has the following form:

$$I = \int_0^1 \frac{x}{\sqrt{\Omega_{r0} + \Omega_{m0}x + \Omega_{\Lambda 0}x^4}} \mathrm{d}x. \tag{35}$$

Since the integral in Eq. (35) is elliptic, following the method in the general case, we obtain the Weierstrass form of the elliptic integral I:

$$I = \int_{s_1}^{s_2} \frac{x_0 + S'(x_0)/\left(4s - S''(x_0)/6\right)}{\sqrt{4s^3 - g_2 s - g_3}} \mathrm{d}s, \tag{36}$$

where:

$$s_1 = \frac{S'(x_0)}{4(1 - x_0)} + \frac{S''(x_0)}{24}, \tag{37}$$

$$s_2 = \frac{-S'(x_0)}{4x_0} + \frac{S''(x_0)}{24}, \tag{38}$$

$$S'(x_0) = \Omega_{m0} + 4\Omega_{\Lambda 0}{x_0}^3, \tag{39}$$

$$S''(x_0) = 12\Omega_{\Lambda 0}{x_0}^2. \tag{40}$$

Here:

$$g_2 = \Omega_{\Lambda 0}\Omega_{r0}, \tag{41}$$

$$g_3 = -\frac{{\Omega_{m0}}^2\Omega_{\Lambda 0}}{16}, \tag{42}$$

are the Weierstrass invariants of the polynomial $S(x) = \Omega_{r0} + \Omega_{m0}x + \Omega_{\Lambda 0}x^4$.
Since the Weierstrass function $\wp(\omega) = \wp(\omega; g_2, g_3)$, attached to g_2 and g_3, satisfies
${\wp'}^2 = 4\wp^3 - g_2\wp - g_3$, by substitution $s = \wp(\omega)$ in Eq. (36) we infer:

$$I = \int_{\wp^{-1}(s_1)}^{\wp^{-1}(s_2)} \left(x_0 + \frac{S'(x_0)}{4\wp(\omega) - S''(x_0)/6} \right) \mathrm{d}\omega. \tag{43}$$

Case $\Omega_{\Lambda 0} = 0$, $\Omega_{r0} > 0$, $\Omega_{m0} > 0$ and $\Omega_{k0} > 0$. This case corresponds to the open universe with mixture of matter and radiation and without the cosmological constant. Since the cosmological constant contribution is neglected, this stage may occurred in the radiation-dominated epoch, or the matter-dominated epoch, or in the transition period between these two epochs, but definitely before domination of the dark energy over radiation. The integral I in Eq. (12) has the following form:

$$I = \int_0^1 \frac{x}{\sqrt{\Omega_{r0} + \Omega_{m0}x + \Omega_{k0}x^2}} \mathrm{d}x. \tag{44}$$

The integral in Eq. (44) is not elliptic, therefore we can find its closed form, which is:

$$
\begin{aligned}
I = &\frac{-2\sqrt{\Omega_{r0}}\sqrt{\Omega_{k0}} + 2\sqrt{\Omega_{k0}}\sqrt{\Omega_{r0}+\Omega_{m0}+\Omega_{k0}}}{2{\Omega_{k0}}^{3/2}} \\
&+ \frac{\Omega_{m0}\ln(\Omega_{m0} + 2\sqrt{\Omega_{r0}}\sqrt{\Omega_{k0}})}{2{\Omega_{k0}}^{3/2}} - \\
&\frac{\Omega_{m0}\ln(\Omega_{m0} + 2\Omega_{k0} + 2\sqrt{\Omega_{k0}}\sqrt{\Omega_{r0}+\Omega_{m0}+\Omega_{k0}})}{2{\Omega_{k0}}^{3/2}}.
\end{aligned}
\tag{45}
$$

By Eq. (5), we have $\Omega_{r0} + \Omega_{m0} + \Omega_{k0} = 1$. Substituting in Eq. (45), we have:

$$
\begin{aligned}
I = &\frac{1-\sqrt{\Omega_{r0}}}{\Omega_{k0}} \\
&+ \frac{\Omega_{m0}}{2{\Omega_{k0}}^{3/2}} \ln\left(\frac{\Omega_{m0} + 2\sqrt{\Omega_{r0}\Omega_{k0}}}{\Omega_{m0} + 2\Omega_{k0} + 2\sqrt{\Omega_{k0}}}\right).
\end{aligned}
\tag{46}
$$

Furthermore, taking for example $\Omega_{m0} = 1 - \Omega_{k0} - \Omega_{r0}$ in Eq. (46), we finally obtain:

$$
\begin{aligned}
I = &\frac{1-\sqrt{\Omega_{r0}}}{\Omega_{k0}} \\
&+ \frac{1-\Omega_{k0}-\Omega_{r0}}{2{\Omega_{k0}}^{3/2}} \ln\left(\frac{1-\sqrt{\Omega_{k0}}+\sqrt{\Omega_{r0}}}{1+\sqrt{\Omega_{k0}}+\sqrt{\Omega_{r0}}}\right).
\end{aligned}
\tag{47}
$$

2.3. Two-Component Universe

Here we consider that two of four constants Ω_{i0} are equal to 0. Therefore, we have six following cases. In all of them, it is assumed that the universe is dominated by two components - the cosmological constant and matter, matter and curvature, matter and radiation, curvature and the cosmological constant, respectively. Some cases in this section are also analyzed in Ryden (2002), where solutions are mostly obtained by some approximations, or are including a new parameter, while our solutions are exact and are depending only on the density parameters.

Case $\Omega_{r0} = 0$, $\Omega_{k0} = 0$, $\Omega_{m0} > 0$ and $\Omega_{\Lambda 0} > 0$. This case corresponds to a pressureless and flat universe with the cosmological constant. Since radiation contribution is neglected, we can say that this is the current stage of evolution of our universe. Note that this period occurred not before the matter-dark energy transition. The integral I in Eq. (12) has the following form:

$$
\begin{aligned}
I &= \int_0^1 \frac{x}{\sqrt{\Omega_{m0}x + \Omega_{\Lambda 0}x^4}}\,\mathrm{d}x \\
&= \frac{2}{3\sqrt{\Omega_{\Lambda 0}}} \ln\left(\frac{\sqrt{\Omega_{\Lambda 0}} + \sqrt{\Omega_{\Lambda 0}+\Omega_{m0}}}{\sqrt{\Omega_{m0}}}\right).
\end{aligned}
\tag{48}
$$

If we take $\Omega_{m0} = \Omega_0$, by Eq. (5) (or setting $t = t_0$ in Eq. (9)) we have $\Omega_0 + \Omega_{\Lambda 0} = 1$. Therefore, the first Carroll-Press-Turner formula (see Carroll et al. 1992) is obtained:

$$
I = \frac{2}{3}\frac{1}{\sqrt{1-\Omega_0}} \ln\left(\frac{1+\sqrt{1-\Omega_0}}{\sqrt{\Omega_0}}\right).
\tag{49}
$$

Case $\Omega_{r0} = 0$, $\Omega_{\Lambda 0} = 0$, $\Omega_{m0} > 0$ and, $\Omega_{k0} > 0$. This case corresponds to a pressureless and open universe without the cosmological constant. At this stage, the universe is matter dominated. The integral I in Eq. (12) has the following form:

$$
\begin{aligned}
I &= \int_0^1 \frac{x}{\sqrt{\Omega_{m0}x + \Omega_{k0}x^2}}\,\mathrm{d}x \\
&= \frac{2\sqrt{\Omega_{k0}}\sqrt{\Omega_{m0}+\Omega_{k0}} + \Omega_{m0}\ln(\Omega_{m0}\Omega_{k0})}{2{\Omega_{k0}}^{3/2}} \\
&- \frac{\Omega_{m0}\ln(\Omega_{k0} + \sqrt{\Omega_{k0}}\sqrt{\Omega_{m0}+\Omega_{k0}})}{{\Omega_{k0}}^{3/2}}.
\end{aligned}
\tag{50}
$$

If we take $\Omega_{m0} = \Omega_0$, by Eq. (5) we have $\Omega_0 + \Omega_{k0} = 1$. Therefore, a variant of the second Carroll-Press-Turner formula (see Carroll et al. 1992) is obtained:

$$
\begin{aligned}
I &= \frac{1}{1-\Omega_0} - \frac{\Omega_0}{(1-\Omega_0)^{3/2}} \sinh^{-1}\left(\sqrt{-1+\Omega_0^{-1}}\right) \\
&= \frac{1}{1-\Omega_0} - \frac{\Omega_0}{(1-\Omega_0)^{3/2}} \ln\left(\frac{1+\sqrt{1-\Omega_0}}{\sqrt{\Omega_0}}\right).
\end{aligned}
\tag{51}
$$

Case $\Omega_{\Lambda 0} = 0$, $\Omega_{k0} = 0$, $\Omega_{m0} > 0$ and $\Omega_{r0} > 0$. This case corresponds to a flat universe with mixture of matter and radiation and without the cosmological constant. Since the cosmological constant contribution is neglected, the situation is similar to the fourth case in the previous section, i.e. this stage may occurred in the radiation-dominated epoch, or matter-dominated epoch, but definitely before the domination of dark energy over radiation. The integral I in Eq. (12) has the following form:

$$
\begin{aligned}
I &= \int_0^1 \frac{x}{\sqrt{\Omega_{r0} + \Omega_{m0}x}}\,\mathrm{d}x \\
&= \frac{4{\Omega_{r0}}^{3/2}}{3{\Omega_{m0}}^2} + \frac{2\left(\Omega_{m0} - 2\Omega_{r0}\right)\sqrt{\Omega_{m0}+\Omega_{r0}}}{3{\Omega_{m0}}^2}.
\end{aligned}
\tag{52}
$$

If we take $\Omega_{m0} = \Omega_0$, by Eq. (5) we have $\Omega_0 + \Omega_{r0} = 1$. Cosequently, we obtain:

$$
I = \frac{4}{3{\Omega_0}^2}\left(\left(1 - {\Omega_0}^{3/2}\right) - 1\right) + \frac{2}{\Omega_0}.
\tag{53}
$$

Case $\Omega_{r0} = 0$, $\Omega_{m0} = 0$, $\Omega_{k0} > 0$ and $\Omega_{\Lambda 0} > 0$. This case corresponds to the open universe with the cosmological constant. Since contributions of matter and radiation are neglected, the universe may be in this stage in the future. The integral I in Eq. (12) has the following form:

$$\begin{aligned} I &= \int_0^1 \frac{x}{\sqrt{\Omega_{k0}x^2 + \Omega_{\Lambda 0}x^4}}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{\Omega_{\Lambda 0}}} \ln\left(\frac{\sqrt{\Omega_{\Lambda 0}} + \sqrt{\Omega_{k0} + \Omega_{\Lambda 0}}}{\sqrt{\Omega_{k0}}}\right). \end{aligned} \tag{54}$$

By Eq. (5) we have $\Omega_{k0} + \Omega_{\Lambda 0} = 1$, and we obtain the expression:

$$I = \frac{1}{\sqrt{1-\Omega_{k0}}} \ln\left(\frac{1+\sqrt{1-\Omega_{k0}}}{\sqrt{\Omega_{k0}}}\right), \tag{55}$$

which is quite similar to the Carroll-Press-Turner formula (see Carroll et al. 1992) in the first mentioned case ($\Omega_{r0} = 0$ and $\Omega_{k0} = 0$).

Case $\Omega_{k0} = 0$, $\Omega_{m0} = 0$, $\Omega_{r0} > 0$ and $\Omega_{\Lambda 0} > 0$. This case corresponds to a radiation dominated and flat universe with the cosmological constant. Since the matter contribution is neglected, the situation is similar to the second case in the previous section, i.e. this stage may occurred in the early universe, since, in some literature, it is stated that the early dark energy may altered the expansion of the universe at that time (see Poulin et al. 2019, Niedermann and Sloth 2019). However, this theory is not concordant with the ΛCDM model, therefore, this case is not possible in the frame of the standard cosmological model, but we will consider it for the sake of completeness of our paper. The integral I in Eq. (12) has the following form:

$$\begin{aligned} I &= \int_0^1 \frac{x}{\sqrt{\Omega_{r0} + \Omega_{\Lambda 0}x^4}}\,\mathrm{d}x \\ &= \frac{1}{2\sqrt{\Omega_{\Lambda 0}}} \ln\left(\frac{\sqrt{\Omega_{\Lambda 0}} + \sqrt{\Omega_{r0} + \Omega_{\Lambda 0}}}{\sqrt{\Omega_{r0}}}\right). \end{aligned} \tag{56}$$

By Eq. (5) we have $\Omega_{\Lambda 0} + \Omega_{r0} = 1$, and so we obtain:

$$I = \frac{1}{2\sqrt{1-\Omega_{r0}}} \ln\left(\frac{1+\sqrt{1-\Omega_{r0}}}{\sqrt{\Omega_{r0}}}\right). \tag{57}$$

Case $\Omega_{\Lambda 0} = 0$, $\Omega_{m0} = 0$, $\Omega_{r0} > 0$ and $\Omega_{k0} > 0$. This case corresponds to a radiation dominated and open universe without the cosmological constant. The integral I in Eq. (12) has the following form:

$$\begin{aligned} I &= \int_0^1 \frac{x}{\sqrt{\Omega_{r0} + \Omega_{k0}x^2}}\,\mathrm{d}x \\ &= \frac{\sqrt{\Omega_{r0} + \Omega_{k0}} - \sqrt{\Omega_{r0}}}{\Omega_{k0}}. \end{aligned} \tag{58}$$

By Eq. (5) we have $\Omega_{r0} + \Omega_{k0} = 1$, and so we obtain:

$$I = \frac{1}{1+\sqrt{\Omega_{r0}}}. \tag{59}$$

2.4. Single-Component Universe

Finally, we assume that three of four constants Ω_{i0} are equal to 0. Therefore, we have four cases, which are the easiest ones to deal with so far. In all of them, it is assumed that the universe is dominated only by one component - the cosmological constant, matter, curvature and radiation, respectively. The cases in this section are also analyzed in Ryden (2002), where the solutions are mostly obtained in terms of the redshift as well as the equation of state, while our solutions are depending only on the density parameters.

Case $\Omega_{r0} = 0$, $\Omega_{m0} = 0$, $\Omega_{k0} = 0$ and $\Omega_{\Lambda 0} > 0$. This case corresponds to the flat universe with the cosmological constant, which is known as the de Sitter universe. The integral I in Eq. (12) has the following form:

$$I = \int_0^1 \frac{x}{\sqrt{\Omega_{\Lambda 0}x^4}}\,\mathrm{d}x = +\infty, \tag{60}$$

wherefrom follows:

$$I = H_0 t_0 = +\infty. \tag{61}$$

Since H_0 is the Hubble constant, we conclude that $t_0 = +\infty$, therefore, this case might happen in the far future.

Case $\Omega_{r0} = 0$, $\Omega_{k0} = 0$, $\Omega_{\Lambda 0} = 0$ and $\Omega_{m0} > 0$. This case corresponds to a matter-dominated, pressureless and flat universe without the cosmological constant, which is known as the Einstein-de Sitter universe. The integral I in Eq. (12) has the following form:

$$I = \int_0^1 \frac{x}{\sqrt{\Omega_{m0}x}}\,\mathrm{d}x = \frac{2}{3\sqrt{\Omega_{m0}}}, \tag{62}$$

wherefrom we obtain:

$$I = \frac{2}{3\sqrt{\Omega_{m0}}}. \tag{63}$$

Case $\Omega_{r0} = 0$, $\Omega_{m0} = 0$, $\Omega_{\Lambda 0} = 0$ and $\Omega_{k0} > 0$. This case corresponds to an open and empty universe without the cosmological constant, which is known as the Milne universe (see Ryden 2002). The integral I in Eq. (12) has the following form:

$$I = \int_0^1 \frac{x}{\sqrt{\Omega_{k0}x^2}}\,\mathrm{d}x = \frac{1}{\sqrt{\Omega_{k0}}}, \tag{64}$$

wherefrom we obtain:

$$I = \frac{1}{\sqrt{\Omega_{k0}}}. \tag{65}$$

It is interesting to consider an "anti-Milne" universe with $\Omega_{k0} < 0$. Since $\Omega_{k0} < 0$, the function in the denominator of the integrand function of the integral I in Eq. (64) is not defined, therefore the integral I in this case does not exist.

Table 1: All Weierstrass and analytical forms for different combinations of cosmological parameters. The analytical solutions do not exist for the first three cases in the Table.

Combination of the cosmological parameters	Weierstrass/analytical form of cosmic time formula
$\Omega_{r0} > 0,\, \Omega_{m0} > 0,\, \Omega_{k0} > 0,\, \Omega_{\Lambda 0} > 0$	$I = \int_{\wp^{-1}(s_1)}^{\wp^{-1}(s_2)} \left(x_0 + \frac{S'(x_0)}{4\wp(\omega) - S''(x_0)/6} \right) \mathrm{d}\omega$
$\Omega_{r0} = 0,\, \Omega_{m0} > 0,\, \Omega_{k0} > 0,\, \Omega_{\Lambda 0} > 0$	$I = \int_{\wp^{-1}(s_1)}^{\wp^{-1}(s_2)} \left(\frac{\Omega_{m0}}{4\wp(\omega) - \Omega_{k0}/3} \right) \mathrm{d}\omega$
$\Omega_{r0} > 0,\, \Omega_{m0} > 0,\, \Omega_{k0} = 0,\, \Omega_{\Lambda 0} > 0$	$I = \int_{\wp^{-1}(s_1)}^{\wp^{-1}(s_2)} \left(x_0 + \frac{S'(x_0)}{4\wp(\omega) - S''(x_0)/6} \right) \mathrm{d}\omega$
$\Omega_{r0} > 0,\, \Omega_{m0} = 0,\, \Omega_{k0} > 0,\, \Omega_{\Lambda 0} > 0$	$I = \frac{1}{2\sqrt{\Omega_{\Lambda 0}}} \ln \left(\frac{1 + \sqrt{\Omega_{\Lambda 0}} + \sqrt{\Omega_{r0}}}{1 - \sqrt{\Omega_{\Lambda 0}} + \sqrt{\Omega_{r0}}} \right)$
$\Omega_{r0} > 0,\, \Omega_{m0} > 0,\, \Omega_{k0} > 0,\, \Omega_{\Lambda 0} = 0$	$I = \frac{1 - \sqrt{\Omega_{r0}}}{\Omega_{k0}} + \frac{1 - \Omega_{k0} - \Omega_{r0}}{2{\Omega_{k0}}^{3/2}} \ln \left(\frac{1 - \sqrt{\Omega_{k0}} + \sqrt{\Omega_{r0}}}{1 + \sqrt{\Omega_{k0}} + \sqrt{\Omega_{r0}}} \right)$
$\Omega_{r0} = 0,\, \Omega_{m0} > 0,\, \Omega_{k0} = 0,\, \Omega_{\Lambda 0} > 0$	$I = \frac{2}{3} \frac{1}{\sqrt{1 - \Omega_0}} \ln \left(\frac{1 + \sqrt{1 - \Omega_0}}{\sqrt{\Omega_0}} \right),\, \Omega_0 = \Omega_{m0}$
$\Omega_{r0} = 0,\, \Omega_{m0} > 0,\, \Omega_{k0} > 0,\, \Omega_{\Lambda 0} = 0$	$I = \frac{1}{1 - \Omega_0} - \frac{\Omega_0}{(1 - \Omega_0)^{3/2}} \ln \left(\frac{1 + \sqrt{1 - \Omega_0}}{\sqrt{\Omega_0}} \right),\, \Omega_0 = \Omega_{m0}$
$\Omega_{r0} > 0,\, \Omega_{m0} > 0,\, \Omega_{k0} = 0,\, \Omega_{\Lambda 0} = 0$	$I = \frac{4}{3{\Omega_0}^2} \left(\left(1 - {\Omega_0}^{3/2} \right) - 1 \right) + \frac{2}{\Omega_0},\, \Omega_0 = \Omega_{m0}$
$\Omega_{r0} = 0,\, \Omega_{m0} = 0,\, \Omega_{k0} > 0,\, \Omega_{\Lambda 0} > 0$	$I = \frac{1}{\sqrt{1 - \Omega_{k0}}} \ln \left(\frac{1 + \sqrt{1 - \Omega_{k0}}}{\sqrt{\Omega_{k0}}} \right)$
$\Omega_{r0} > 0,\, \Omega_{m0} = 0,\, \Omega_{k0} = 0,\, \Omega_{\Lambda 0} > 0$	$I = \frac{1}{2\sqrt{1 - \Omega_{r0}}} \ln \left(\frac{1 + \sqrt{1 - \Omega_{r0}}}{\sqrt{\Omega_{r0}}} \right)$
$\Omega_{r0} > 0,\, \Omega_{m0} = 0,\, \Omega_{k0} > 0,\, \Omega_{\Lambda 0} = 0$	$I = \frac{1}{1 + \sqrt{\Omega_{r0}}}$
$\Omega_{r0} = 0,\, \Omega_{m0} = 0,\, \Omega_{k0} = 0,\, \Omega_{\Lambda 0} > 0$	$I = +\infty$
$\Omega_{r0} = 0,\, \Omega_{m0} > 0,\, \Omega_{k0} = 0,\, \Omega_{\Lambda 0} = 0$	$I = \frac{2}{3\sqrt{\Omega_{m0}}}$
$\Omega_{r0} = 0,\, \Omega_{m0} = 0,\, \Omega_{k0} > 0,\, \Omega_{\Lambda 0} = 0$	$I = \frac{1}{\sqrt{\Omega_{k0}}}$
$\Omega_{r0} > 0,\, \Omega_{m0} = 0,\, \Omega_{k0} = 0,\, \Omega_{\Lambda 0} = 0$	$I = \frac{1}{2\sqrt{\Omega_{r0}}}$

Case $\Omega_{m0} = 0$, $\Omega_{k0} = 0$, $\Omega_{\Lambda 0} = 0$ and $\Omega_{r0} > 0$. This case corresponds to a radiation dominated and flat universe without the cosmological constant. The integral I in Eq. (12) has the following form:

$$I = \int_0^1 \frac{x}{\sqrt{\Omega_{r0}}}\,\mathrm{d}x = \frac{1}{2\sqrt{\Omega_{r0}}}, \tag{66}$$

wherefrom we obtain:

$$I = \frac{1}{2\sqrt{\Omega_{r0}}}. \tag{67}$$

3. CONCLUSION

The Weierstrass canonical form of the cosmic time formula is obtained in the case of four-component universe obeying the ΛCDM model, as well as in two cases for the three-component universe, with $\Omega_{r0} = 0$ and $\Omega_{k0} = 0$. In all other cases the analytical solutions of the integral I are obtained. All Weierstrass forms, as well as all possible cases that concern analytical solution of the integral I for different combinations of the cosmological parameters are presented in Table 1. We note that every choice of dominant components in the universe has its physical interpretation, which is concordant with the appropriate stage in the universe's evolution. Having that in mind, we improved the cases that are discussed in various literature and united them with the new ones that are presented in this paper in order to obtain all phases of the universe, from its beginning. Also, we believe that we found new analytical solutions for the three-component universe with $\Omega_{\Lambda 0} = 0$ and for the two component universe with $\Omega_{\Lambda 0} = 0$, $\Omega_{k0} = 0$, which may correspond to the early stage in the evolution of the universe.

We believe that this paper is useful not only for improving and summarizing already known results with two new analytical solutions, but also for observing evolution of the universe from perspective of theory of elliptic functions, which turned out to be quite fruitful when it comes to applications in cosmology.

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КОСМИЧКО ВРЕМЕ ЗА УНИВЕРЗУМ КОЈИ СЕ САСТОЈИ ОД ВИШЕ КОМПОНЕНАТА

D. Branković

Department of Applied Mathematics, School of Electrical Engineering, University of Belgrade
Bulevar kralja Aleksandra 73, 11000 Belgrade, Serbia

E–mail: *danijela@etf.bg.ac.rs*

УДК 524.834
Оригинални научни рад

У случају ΛCDM модела универзума који се састоји од четири компоненте, као и у неколико случајева универзума који се састоји од три компоненте, одредили смо Вајерштрасову канонску форму за формулу која одређује космичко време. У свим другим случајевима, дискутована су аналитичка решења за формулу која одређује космичко време.