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DEFLECTION OF LIGHT IN EQUATORIAL PLANE DUE TO KERR-TAUB-NUT BODY

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SUMMARY: According to General Relativity, there are factors like mass, rotation, charge and presence of Cosmological constant that can influence the path of light ray. Apart from these factors, many authors have also reported the influence of gravitomagnetism on the path of light ray. In this study we have discussed the effect of a rotating Kerr-Taub-NUT body where the strength of the gravitomagnetic monopole is represented by the NUT factor or magnetic mass. We use the null geodesic of photon method to obtain the deflection angle of light ray for a Kerr-Taub-NUT body in equatorial plane upto the fourth order term. Our study shows that the NUT factor has a noticeable effect on the path of the light ray. By considering the magnetism to be zero, the expression of bending angle gets reduced to the Kerr bending angle. However, we obtained a non-zero bending angle for a hypothetical massless, magnetic body.

Key words. Gravitation – Gravitational lensing – Relativity

1. INTRODUCTION

One of the most significant predictions of general relativity is gravitational light deflection. There are few physical parameters that can affect the spacetime geometry, namely, gravitational mass, rotation, charge and cosmological constant. Einstein calculated the first order contribution of mass on the path of light ray. After Einstein, a number of authors worked in this field and obtained the higher order contribution of mass for static bodies (Keeton and Petters 2005, Virbhadra and Ellis 2000, Iyer and Petter 2007) and naked singularity (Virbhadra and Ellis 2002) for both strong and weak field limit. On the other hand, few authors were working on rotating Kerr mass in equatorial (Iyer and Hansen 2009) or off equatorial plane (Bozza 2003, Chakraborty and Sen 2015a) and obtained the deflection angle as well as other lensing parameters. Azzami et al. (2011) obtained the two individual components (parallel and perpendicular to equatorial plane) of the light deflection angle in quasi-equatorial regime. In one of the studies, Dubey and Sen (2014, 2015) have used the Kerr and Kerr-Newman mass to show how gravitational redshift gets affect as a photon is emitted from various latitudes. In some recently published work, the present authors (Chakraborty and Sen 2015b) and (Hasse and Pelrick 2006) showed the effect of rotating charge bodies on the path of light ray. Eiroa et al. (2002) calculated the deflection angle and lensing parameters for static charge bodies. Virbhadra et al. (1998) worked on the Janis-Newman-Winicour (JNW) mass which is a charged, static mass and calculated the light deflection angle up to the second order.

On the other hand, some authors have used the material medium approach, where the gravitational effect on light ray was calculated by assuming some effective refractive index assigned to the medium through which light propagates. Using this approach Atkinson (1965) studied the trajectory of light ray near an extremely massive, static and spherically symmetric star. This approach was also used to calculate deflection angle of light, upto second order Fischback and Freeman (1982) and under strong deflection limit A. K. Sen (2010) for static bodies. In earlier past a similar method was used by Balaz (1958), to calculate the change in the direction of polarization vector of electromagnetic wave passing close to a rotating body. Roy and Sen (2015) in one of their recent works have calculated the trajectory of a light ray in Kerr field using this material medium approach.

The concept of the effect of cosmological constant on the light deflection angle was first introduced by Rindler and Ishak (2007). They derived the formula for light deflection angle under de Sitter back ground (i.e the non zero cosmological constant) according to which a positive cosmological constant diminishes the bending angle. Bhadra et al. (2010) obtained the light deflection angle in the Schwarzschild de Sitter (SDS) geometry, where they computed the bending angle in the SDS space-time by taking the proper cosmological constant involved in the solution of the trajectory of light. Sultana (2013) in his paper used the method of Rindler and Ishak (2007) to obtain a weak field approximation for the bending angle in the Kerr de Sitter space time. Kraniotis (2014) obtained the solution of null geodesics that describes photon orbits in the space time of a rotating electrically charged black hole including the contribution from the cosmological constant.

The concept of generalized Schwarzschild metric was first introduced by Newman, Tamburino and Unti (1963). This metric contains one arbitrary parameter in addition to the mass generally known as the NUT factor or gravitomagnetic mass. In the same year, Misner (1963) studied the generalized Schwarzschild metric and called it as NUT (named after Newman, Tamburino and Unti) space time. According to him, this line element has a Schwarzschild-like singularity, but this singularity is not observed in the curvature tensor. The presence of the cross term $dtd\varphi$, shows that this space has a strength of gravitomagnetic monopole (Nouri-Zonoz and Lynden-Bell 1998). Lensing effect of this type of mass was studied by Nouri-Zonoz and Lynden-Bell (1997). Chakraborty and Sen (2017) have obtained the deflection angle due to Taub-NUT body and showed that the NUT factor and static charge have complete opposite effect to the space-time geometry.

The Kerr-Taub-NUT (KTN) line element (Demianski and Newman 1966, Miller 1973) contains three parameters, namely the mass, rotation parameter, and NUT factor. Wei et al. (2012) numerically studied the quasi-equatorial lensing by the stationary, axially-symmetric black hole in the KTN space time in the strong field limit. Abdujabbarov et al. (2008) studied the electromagnetic fields in the KTN space time as well as in the surrounding space time of a slowly rotating magnetized NUT star and obtained analytical solutions of Maxwell equations. Chakraborty and Majumdar (2014) derived the exact Lense Thirring precession frequencies for the Kerr, KTN and Taub-NUT space time. Pradhan (2015) performed a detailed analysis of photon orbit by investigating the equatorial null circular geodesics as well as the time like geodesic for the Kerr-Newman-Taub-NUT black hole. He obtained the conditions for the existence of marginally bound circular orbit and null circular geodesics of the Kerr-Newman-Taub-NUT space times. Cebeci et al. (2016) used the Hamilton-Jacobi method to derive the equations of motion for a charged test particle in the background of the Kerr-Newman-Taub-NUT spacetime. They also examined the stability of spherical orbits with respect to the NUT parameter. For zero static charge, results in Pradhan (2015) and Cebeci et al. (2016) are reduced to that of the KTN space time.

In this present work, we studied the KTN line element and obtained the equatorial light deflection angle for such space time geometry up to fourth order term which is a function of mass, rotation parameter and the NUT factor. We also studied variation of the light deflection angle as a function of the NUT factor. For zero NUT factor our result reduces to the well known Kerr light deflection angle.

2. GENERAL GEODESICS EQUATIONS IN KTN SPACE TIME

The KTN solution expressed in the Boyer-Lindquist-like coordinates $(ct, r, \vartheta, \varphi)$ is given by the following metric (Demianski and Newman 1966, Miller 1973),

$$ds^2 = \frac{-\Delta}{\rho^2}[cdt + (2n\cos\vartheta - a\sin^2\vartheta)d\varphi]^2 + \frac{\sin^2\vartheta}{\rho^2}[acdt - (r^2 + n^2 + a^2)d\varphi]^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\vartheta\,, \tag{1}$$

where, $\Delta = r^2 - 2mr + a^2 - n^2$, $\rho^2 = r^2 + (n + a\cos\vartheta)^2$, $m = \frac{GM}{c^2}$ and $a = \frac{J}{cM}$ further c, G, M, J and n are the velocity of light in free space, gravitational constant, mass, angular momentum of the gravitating body, and the NUT charge. Both m and n have the dimension of length. If we set $a = 0$, Eq. (1) will reduce to Taub-NUT solution.

Two horizons are located at the of roots of $r^2 - 2mr + a^2 - n^2 = 0$ i.e:

$$r_\pm = m \pm \sqrt{m^2 - a^2 + n^2}\,. \tag{2}$$

As shown by Miller (1973), the KTN space time does not have curvature singularities but there exist conical singularities on the axis of symmetry which lead to the emergence of closed time-like curves in the space. So it can be concluded that, the KTN solution is completely different from other rotating solutions of Einstein's equation in terms of singularity structure (Cebei et al. 2016).

Now that we have considered that the light ray is moving through the KTN space time, the related generalized geodesic equations can be obtained from the line element in Eq. (1) using the Hamilton-Jacobi equation (Chandrasekhar 1983, Bini et al. 2003) as:

$$\rho^2 \dot{r} = \sqrt{R(r)}\,, \tag{3}$$

$$\rho^2 \dot{\vartheta} = \sqrt{\Theta(\vartheta)}\,, \tag{4}$$

$$\rho^2 \dot{\varphi} = -[a - 2n\cot\vartheta\csc\vartheta]E + L\csc^2\vartheta + \frac{a[E(r^2+n^2+a^2) - aL]}{\Delta}\,, \tag{5}$$

$$\begin{aligned}\rho^2 c\dot{t} &= -E[\sin^2\vartheta(a + 2n\csc^2\vartheta)^2 - 4n\{n + a(1+\cos\vartheta)\}]\\ &-(2n\cos\vartheta\csc^2\vartheta - a)L + \frac{(r^2+a^2+n^2)}{\Delta}[E(r^2+a^2+n^2) - aL]\,,\end{aligned} \tag{6}$$

where:

$$R(r) = [E(r^2+a^2) - aL]^2 + n^2E[2a^2E - 2aL + E(n^2+2r^2)] - \Delta[K + (aE-L)^2] \tag{7}$$

and:

$$\Theta(\vartheta) = K - \cos^2\vartheta[-a^2E^2 + \frac{L^2}{\sin^2\vartheta}] + 2n\cos\vartheta[2aE^2 + 2EL\csc^2\vartheta] - 4n^2E^2\cot^2\vartheta\,. \tag{8}$$

Here the dot indicates the derivative with respect to the affine parameter. The three constants of motion are L (angular momentum of the particle along the direction of rotation), E (energy of the particle), and K (Carter constant).

3. EQUATORIAL GEODESICS EQUATIONS IN KTN SPACE TIME

The main objective of this paper is to calculate the light deflection angle for the KTN geometry in the equatorial plane. So, at this point we converted the generalized geodesic equations to equatorial geodesic equations. The conditions for the equatorial plane are $\vartheta = \frac{\pi}{2}$ (Pradhan 2015). But Cebeci et al. (2016) showed that equatorial orbits can exist for arbitrary NUT parameter for the constraint relation between energy, angular momentum of test particle, and the rotation parameter which is:

$$L = \frac{a(2E^2 - \mu^2)}{2E}\,,$$

where μ is the mass of the test particle. Using these conditions, the modified equatorial geodesic equations can be obtained.

The modified version of Eq. (3) will be:

$$(r^2+n^2)^2\dot{r}^2 = [E(r^2+a^2) - aL]^2 + n^2E[2a^2E - 2aL + E(n^2+2r^2)] - \Delta(aE-L)^2\,. \tag{9}$$

Here the dot indicates the derivative with respect to the affine parameter τ. As we have already shown in our previous work (Chakraborty and Sen 2015b) that the impact parameter is the ratio of L and E (Chandrasekhar 1983). Following Iyer and Hansen (2009), we write the impact parameter $b_s \equiv sb = s(\frac{L}{E})$, where $s = +1$ for a prograde and $s = -1$ for a retrograde orbit of light ray and b is the positive magnitude of the impact parameter. So, the new form of the above equation is:

$$\begin{aligned}(r^2+n^2)^2\dot{r}^2 &= L^2[\frac{r^4}{b^2} + (1 - \frac{a}{b_s})^2(2mr + n^2) - r^2(1 - \frac{a^2}{b^2})\\ &+2n^2\{(1 - \frac{a}{b_s}) - (1 - \frac{a^2}{b^2})\} + \frac{n^2}{b^2}(4r^2 + n^2)]\,.\end{aligned} \tag{10}$$

As r obtains a local extremum for the closest approach r_0, we write:

$$\dot{r}|_{r=r_0} = 0\,.$$

Thus, from Eq. (10):

$$\frac{r_0^2}{b^2} = -(1-\frac{a}{b_s})^2(2\frac{m}{r_0}+\frac{n^2}{r_0^2}) + (1-\frac{a^2}{b^2})$$
$$-2\frac{n^2}{r_0^2}\{(1-\frac{a}{b_s}) - (1-\frac{a^2}{b^2})\} - \frac{n^2}{b^2}(4+\frac{n^2}{r_0^2})\,. \tag{11}$$

From Eqs. (4) and (8), it is clear that:

$$\dot{\vartheta}|_{\vartheta=\frac{\pi}{2}} = 0\,. \tag{12}$$

From Eq. (5) we get (under boundary condition $\vartheta = \frac{\pi}{2}$):

$$(r^2+n^2)\dot{\varphi} = \frac{L}{\Delta}[\frac{2mra}{b} + r^2 - 2mr + n^2(\frac{2a}{b}-1)]\,. \tag{13}$$

From Eq. (6) we get (under boundary condition $\vartheta = \frac{\pi}{2}$):

$$(r^2+n^2)c\dot{t} = \frac{1}{\Delta}[-E\{a^2(r^2-2mr+a^2-n^2) - (r^2+a^2+n^2)^2\} - 2aL(mr+n^2)]\,. \tag{14}$$

Eqs. (10), (13) and (14) are the equatorial geodesic equations governing the motion of light ray under the KTN space time geometry.

4. RADIUS OF PHOTON SPHERE

In this section of the paper we obtain the equatorial circular orbit (representing photon sphere) with radius $r_{\rm ph}$. The photon radius $r_{\rm ph}$ is defined by the condition $R(r) = \frac{dR(r)}{dr} = 0$, for $r = r_{\rm ph}$ (Chakraborty and Sen 2015). So, from Eq. (9) we have the following equations:

$$E^2(r^4+n^4) + (L-aE)^2(2mr+n^2) - r^2(L^2-a^2E^2) + n^2[2a^2E^2 - 2aEL + 2r^2E^2] = 0$$

or:

$$E^2 + (\frac{2m}{r_{\rm ph}^3} + \frac{n^2}{r_{\rm ph}^4})(L-aE)^2 - \frac{1}{r_{\rm ph}^2}(L^2-a^2E^2) + E^2(\frac{n^4}{r_{\rm ph}^4})$$
$$+\frac{n^2}{r_{\rm ph}^4}[2a^2E^2 - 2aEL] + \frac{2n^2E^2}{r_{\rm ph}^2} = 0\,,$$

and:

$$\frac{d}{dr}[E^2 + (\frac{2m}{r_{\rm ph}^3} + \frac{n^2}{r_{\rm ph}^4})(L-aE)^2 - \frac{1}{r_{\rm ph}^2}(L^2-a^2E^2) + E^2(\frac{n^4}{r_{\rm ph}^4})$$
$$+\frac{n^2}{r_{\rm ph}^4}[2a^2E^2 - 2aEL] + \frac{2n^2E^2}{r_{\rm ph}^2}] = 0\,,$$

or:

$$r_{\rm ph}^2\{(L^2-a^2E^2) - 2n^2E^2\} - r_{\rm ph}\{3m(L-aE)^2\} - \{2n^2(L-aE)^2 + 2E^2n^4 + 2n^2(2a^2E^2 - 2aEL)\} = 0\,.$$

Solving the above equation for $r_{\rm ph}$ we get,

$$r_{\rm ph} = [3m(L-aE)^2 \pm [9m^2(L-aE)^2 - 4n^2\{(L^2-a^2E^2) - 2n^2E^2\}\{2(L-aE)^2$$
$$+2E^2(n^2+a^2) - 2aEL\}]^{\frac{1}{2}}]/[2\{(L^2-a^2E^2) - 2n^2E^2\}]\,.$$

The above equation represents the equatorial circular orbit. For $n = 0$, we get:

$$r_{\rm ph} = 3m\frac{L-aE}{L+aE}\,.$$

This is the expression for radius of equatorial circular orbit (Chandrasekhar 1983).

5. EQUATORIAL LIGHT DEFLECTION ANGLE

The light deflection angle can be expressed as (Weinberg 1972):

$$\alpha = 2\int_{r_0}^{\infty}(\frac{d\varphi}{dr}).dr - \pi\,. \tag{15}$$

Now, using Eqs. (10) and (13) in Eq. (15) we have:

$$\begin{aligned}\alpha = 2\int_{r_0}^{\infty}[\frac{2mra}{b} + r^2 - 2mr + n^2(\frac{2a}{b} - 1)]/\Delta[\frac{r^4}{b^2} + (1-\frac{a}{b_s})^2(2mr+n^2)\\ -r^2(1-\frac{a^2}{b^2}) + 2n^2\{(1-\frac{a}{b_s}) - (1-\frac{a^2}{b^2})\} + \frac{n^2}{b^2}(4r^2+n^2)]^{\frac{1}{2}}.dr - \pi\end{aligned} \tag{16}$$

or:

$$\begin{aligned}\alpha = 2\int_{r_0}^{\infty}[1 - \frac{2m}{r} + \frac{2ma}{br} + \frac{n^2}{r^2}(\frac{2a}{b} - 1)]/[1 - \frac{2m}{r} + \frac{a^2}{r^2} - \frac{n^2}{r^2}][\frac{r^4}{b^2} + (1-\frac{a}{b_s})^2(2mr+n^2)\\ -r^2(1-\frac{a^2}{b^2}) + 2n^2\{(1-\frac{a}{b_s}) - (1-\frac{a^2}{b^2})\} + \frac{n^2}{b^2}(4r^2+n^2)]^{\frac{1}{2}}.dr - \pi\,.\end{aligned} \tag{17}$$

We introduce a new variable $x = \frac{r_0}{r}$. So:

$$dx = -\frac{r_0 dr}{r^2}$$

or:

$$\frac{dx}{r_0} = -\frac{dr}{r^2}\,.$$

The limits change as: when $r \longrightarrow \infty$, then $x \longrightarrow 0$, and when $r \longrightarrow r_0$, then $x \longrightarrow 1$. We use this in above equation:

$$\alpha = 2\int_0^1 \frac{f_1}{f_2\sqrt{f_3}}.dx - \pi\,, \tag{18}$$

where:

$$f_1 = 1 - 2hxF - l^2x^2(2F-1)$$
$$f_2 = 1 - 2hx + \hat{a}^2h^2x^2 - l^2x^2\,,$$

and:

$$f_3 = \frac{r_0^2}{b^2} + F^2x^2(2hx + l^2x^2) - Gx^2 + 2l^2x^4(F-G) + \hat{l}^2x^2(4 + l^2x^2)\,.$$

Further, $h = \frac{m}{r_0}$ and $l^2 = \frac{n^2}{r_0^2}$, $\hat{l}^2 = \frac{n^2}{b^2}$, and $\hat{a} = \frac{a}{m}$. So, the mass and NUT factor are now represented by h and l. We again follow Aazami et al. (2011) and substitute $G = 1-(\frac{a}{b})^2 = 1-\hat{a}^2(\frac{m}{b})^2$ and $F = 1-(\frac{a}{b_s}) = 1-s\hat{a}\frac{m}{b}$. Thus for zero rotation ($\hat{a} = 0$), $F = G = 1$.

We use the expression:

$$\begin{gathered}\frac{r_0^2}{b^2} = -(1-\frac{a}{b_s})^2(2\frac{m}{r_0} + \frac{n^2}{r_0^2}) + (1-\frac{a^2}{b^2}) - 2\frac{n^2}{r_0^2}\{(1-\frac{a}{b_s}) - (1-\frac{a^2}{b^2})\} - \frac{n^2}{b^2}(4 + \frac{n^2}{r_0^2}) =\\ G - F^2(2h + l^2) - 2l^2(F-G) - \hat{l}^2(4 + l^2)\end{gathered}$$

from Eq. (11) in f_3 and after rearranging we get:

$$f_3 = (G - 4\hat{l}^2)(1-x^2) - 2hF^2(1-x^3) - [F^2l^2 + 2l^2(F-G) + \hat{l}^2l^2](1-x^4)\,.$$

We substitute $G - 4\hat{l}^2 = A$, so the new form of f_3 is:

$$f_3 = A(1-x^2) - 2hF^2(1-x^3) - [F^2l^2 + 2l^2(F-G) + \hat{l}^2l^2](1-x^4)\,,$$

or:

$$f_3 = A(1-x^2)[1-\frac{2hF^2}{A}(\frac{1-x^3}{1-x^2})][1-\frac{\{F^2l^2+2l^2(F-G)+\hat{l}^2l^2\}}{A}(1+x^2)\{1-\frac{2hF^2}{A}(\frac{1-x^3}{1-x^2})\}^{-1}]\,.$$

Following Chakraborty and Sen (2015b), we substitute $\frac{F^2h(1-x^3)}{A(1-x^2)} = \delta_2$ and $\frac{\{F^2l^2+2l^2(F-G)\}(1+x^2)}{A} = \delta_1$. So, the new form of f_3, using δ_2 and δ_1 is:

$$f_3 = A(1-x^2)[1-2\delta_2][1-(\delta_1+\frac{l^2\hat{l}^2(1+x^2)}{A})(1-2\delta_2)^{-1}]\,.$$

Putting these values of f_1, f_2 and f_3 in Eq. (18), we get:

$$\alpha = 2\int_0^1 \frac{f_1 f_2^{-1}}{\sqrt{A}\sqrt{1-x^2}\sqrt{1-2\delta_2}\sqrt{1-(\delta_1+\frac{l^2\hat{l}^2(1+x^2)}{A})(1-2\delta_2)^{-1}}}.dx - \pi\,. \tag{19}$$

Now, rearranging the above equation we write:

$$\alpha = 2\int_0^1 \frac{dx}{\sqrt{A}\sqrt{1-x^2}} f_1 f_2^{-1}(1-2\delta_2)^{-\frac{1}{2}}[1-(\delta_1+\frac{l^2\hat{l}^2(1+x^2)}{A})(1-2\delta_2)^{-1}]^{-\frac{1}{2}} - \pi\,. \tag{20}$$

For the weak deflection limit, one can assume $n, m \ll r_0$, in other words, $h, l \ll 1$. So, Eq. (18) can be expanded in the Taylor series in terms of both h and l. Here we calculate the deflection angle considering contribution up to fourth order terms in the mass and NUT factor only and write:

$$\alpha = 2\int_0^1 \frac{dx}{\sqrt{A}\sqrt{1-x^2}} f_1[1+2hx+x^2h^2(4-\hat{a}^2)+x^3h^3(8-4\hat{a}^2)+x^4h^4(\hat{a}^4-12\hat{a}^2+16)]$$
$$[1+x^2l^2(1+2hx+x^2h^2\{4-\hat{a}^2\})+x^4l^4][1+\delta_2+\frac{3}{2}\delta_2^2+\frac{5}{2}\delta_2^3+\frac{35}{8}\delta_2^4]$$
$$[1+\frac{\delta_1}{2}(1+2\delta_2+4\delta_2^2)+\frac{3}{8}\delta_1^2+\frac{l^2\hat{l}^2(1+x^2)}{2A}] - \pi\,.$$

In above, we substitute $s_0 = -\hat{a}^2+4-4F$, $s_1 = -4\hat{a}^2+8+2F\hat{a}^2-8F$ and $s_2 = \hat{a}^4-12\hat{a}^2+16+8\hat{a}^2F-16F$ by following Chakraborty and Sen (2015b). Then, multiplying and integrating term by term and retaining only up to fourth order in both h and $l, \hat{l}$ we get the new form the above equation as:

$$\alpha = c_0\pi + 4h[c_1 - \frac{l^2(1-F)}{\sqrt{A}}\{2+(\frac{8}{3}-\frac{\pi}{2})\frac{F^2}{A}+\frac{5}{6A}(F^2+2(F-G))\}+(\frac{7}{2}-\frac{3\pi}{8})\frac{l^2F^2(F^2+2(F-G))}{A^{\frac{5}{2}}}]$$
$$+h^2[-4c_2+\frac{15\pi}{4}d_2+\frac{l^2}{\sqrt{A}}\{\frac{3\pi}{8}(16-16F-3\hat{a}^2+2\hat{a}^2F)+(-24+\frac{45\pi}{4})\frac{F^2(1-F)}{A}+\frac{F^4(1-F)}{A^2}(\frac{105\pi}{8}-32)$$
$$+\frac{7\pi s_0}{16A}(F^2+2(F-G))+(\frac{825\pi}{32}-50)\frac{F^4(F^2+2(F-G))}{A^3}+(\frac{81\pi}{8}-18)\frac{F^2(1-F)(F^2+2(F-G))}{A^2}\}]$$
$$+h^3\Big(\frac{122}{3}c_3-\frac{15\pi}{2}d_3\Big)+h^4\Big(-130c_4+\frac{3465\pi}{64}d_4\Big)+l^2[\frac{(1-F)\pi}{\sqrt{A}}+\frac{3\pi\{(F^2+2(F-G))+\hat{l}^2\}}{4A^{\frac{3}{2}}}]$$
$$+l^4[\frac{3(1-F)\pi}{4\sqrt{A}}+\frac{7\pi(1-F)}{8A^{\frac{3}{2}}}(F^2+2(F-G))+\frac{57\pi(F^2+2(F-G))^2}{64A^{\frac{5}{2}}}]\,, \tag{21}$$

where:

$$c_0 = \frac{1}{\sqrt{A}} - 1\,,$$

$$c_1 = \frac{F^2+A-FA}{A^{\frac{3}{2}}}\,,$$

$$c_2 = \frac{F^2}{A} c_1 \,,$$

$$d_2 = \frac{1}{15A^{\frac{5}{2}}} [15F^4 - 4A(F-1)(3F^2 + 2A) - 2A^2 \hat{a}^2] \,,$$

$$c_3 = \frac{1}{61A^{\frac{7}{2}}} [61F^6 - A(F-1)(45F^4 + 32F^2A + 16A^2) - 4G^2\hat{a}^2(2F^2 + 2A - FA)] \,,$$

$$d_3 = \frac{F^2}{A} d_2 \,,$$

$$c_4 = \frac{F^2}{65A^{\frac{9}{2}}} [65F^6 - 49(F-1)F^4A + 8F^2A^2s_0 + 2s_1A^3] \,,$$

$$d_4 = \frac{1}{1155A^{\frac{9}{2}}} [1155F^8 - 840(F-1)F^6A + 140F^4s_0A^2 + 40s_1F^2A^3 + 8s_2A^4] \,.$$

The above substitution was followed from Aazami et al. (2011) and Chakraborty and Sen (2015b).

The above Eq. (21) represents the equatorial deflection of light under the KTN geometry in the weak field limit. This expression is a function of mass, rotation and the NUT factor. As mentioned earlier in the text, the assumption used for derivation here is that both $n \ll r_0$ and $m \ll r_0$. Thus, this result can be reduced to that of the Kerr metric by setting $n = 0$.

If we set the NUT factor equal to zero in Eq. (21), we can have the expression of deflection for light by the Kerr mass obtained by Aazami et al. (2011).

If we set mass and rotation equal to zero, i.e $h = 0$, $F = G = 1$ and $A = 1 - 4\hat{l}^2$ in Eq. (21), we get:

$$\alpha = l^2 [\frac{3\pi}{4} \frac{[1 + \hat{l}^2]}{(1 - 4\hat{l}^2)^{\frac{3}{2}}}] + l^4 [\frac{57\pi}{64(1 - 4\hat{l}^2)^{\frac{5}{2}}}] \,. \tag{22}$$

This is the amount of deflection of light ray occurring only due to the NUT factor. Thus, a hypothetical massless, static body with non zero NUT factor can influence the curvature of space time.

6. DISCUSSION OF RESULTS

In this study, we obtain the expression of the event horizon from Eq. (2) which is $r_{\pm} = m \pm \sqrt{m^2 - a^2 + n^2}$. The above expression represents two event horizons and for $a^2 > m^2 + n^2$, there will be no event horizon, i.e. naked singularity will appear, which will violate the causality of the space time and is forbidden according to the Penroses cosmic censorship conjecture (Wei et al. 2012). It clearly gives a limit to the value of $(m^2 + n^2)$, i.e it must be greater than a^2.

To understand the physical significance of the calculation done in this paper, we plot the bending angle (α) against various physical parameters ($\hat{a}$, l, b) in Fig. 1, Fig. 2, Fig. 3, respectively by taking the Sun as a test case. In our previous work (Chakraborty and Sen 2015b), we obtained the equatorial deflection angle for the Kerr-Newman (rotation with electric charge) body where we used the Sun as a test case and plotted bending angle against various physical parameters by considering that the Sun has some static charge and the closest approach of the light ray is the radius of the Sun. Following Chakraborty and Sen (2015b), here we consider that the Sun has some NUT charge $l = 1.413850947 \times 10^{-6}$ and the closest approach

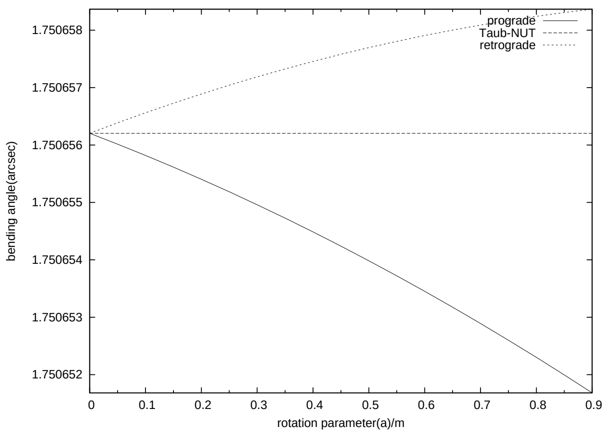

Fig. 1. *Bending angle (arcsec) as a function of rotation parameter (a/m) with constant NUT factor* $l = 1.413850947 \times 10^{-6}$ *and the impact parameter is of one solar radius. Here, three different curves represent the prograde (when the light ray moves in the direction of rotation of the body), corresponding Taub-NUT (zero rotation), and retrograde (when the light ray moves in the opposite direction of the rotation of the body) motion of light ray.*

is the radius of the Sun (6.955×10^8 m). The value of $l = n/r_0$ was chosen in such a way that $(m^2 + n^2)$ is always greater than a^2. Here we would like to mention that the values of the NUT parameter are chosen arbitrarily as we do not know the exact value of the NUT parameter related to any astrophysical body. But we took the values of l in the same order as that of the static charge parameter of our previous work (Chakraborty and Sen 2015) to draw a comparison between the effect of the NUT factor and static charge on the space-time geometry.

Fig. 1 clearly shows the difference between the prograde and retrograde motion with respect to the zero rotation Taub-NUT case. The nature of bending angle versus the rotation parameter curve is similar to the result obtained by Iyer and Hansen (2009) and Chakraborty and Sen (2015b) for the Kerr and Kerr-Newman equatorial bending, respectively.

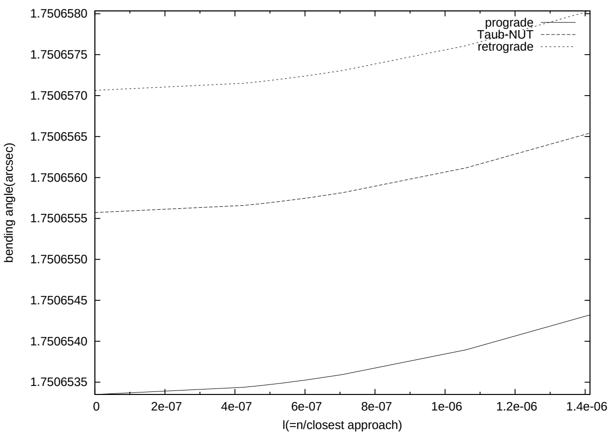

Fig. 2. *Bending angle (arcsec) as a function of charge $l = n/r_o$ with constant rotation parameter $\hat{a} = 0.5$ and impact parameter is one solar radius. Geometries are explained in the caption for Fig. 1.*

From Fig. 2, it can be said that deflection of the light ray increases with the increase of the NUT parameter of the body. For the zero NUT factor, the light ray has minimum deflection. We know that the presence of static charge reduces the amount of deflection of light (Chakraborty and Sen 2015b). So, it can be seen that the NUT factor and static charge influence the space time geometry in opposite direction. The presence of the NUT factor increases the light deflection angle compared to the zero field Kerr case. On the other hand, the presence of static charge decreases the light deflection angle with respect to the Kerr case.

Fig. 3 shows the change of bending angle with impact parameter, though the pattern is similar to that given by Iyer and Hansen (2009) and Chakraborty and Sen (2015b), but the prograde, retrograde, and the zero spin Taub-NUT plot overlap with each other as the difference between them are small for the Sun. So, following Chakraborty and Sen (2015b), we consider a slow rotating pulsar PSR J 1748-2446 (Nuñez and Nowakowski 2010) as a test case and plot the bending angle against the impact parameter in Fig. 4. We consider that the pulsar has some NUT charge $l = 1.413850947 \times 10^{-6}$ and that the closest approach corresponds to the physical radius of the pulsar which is 20 km. The pattern of the plot is similar to that given by Iyer and Hansen (2009) and Chakraborty and Sen (2015b). For this pulsar we consider $m = 1.99$ km and calculate a as 0.96 km from the input values of time period $T = 1.393$ ms and $r_0 = 20$ km as listed by Nuñez and Nowakowski (2010).

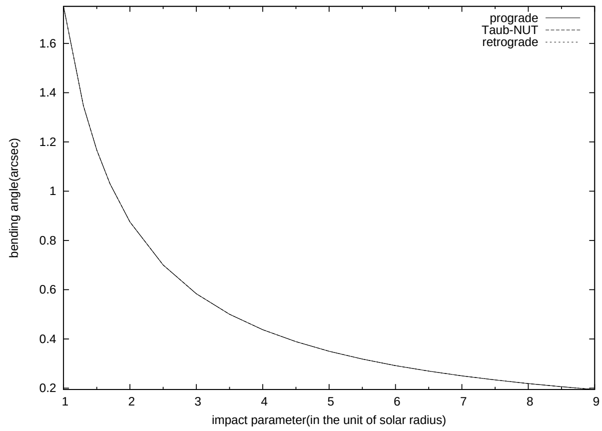

Fig. 3. *Bending angle (arcsec) as a function of impact parameter in the unit of solar radius with constant rotation ($\hat{a} = 0.5$) and the NUT factor $l = 1.413850947 \times 10^{-6}$. Geometries are explained in the caption for Fig. 1. All the plots for the prograde, Taub-NUT and retrograde merge together for the Sun.*

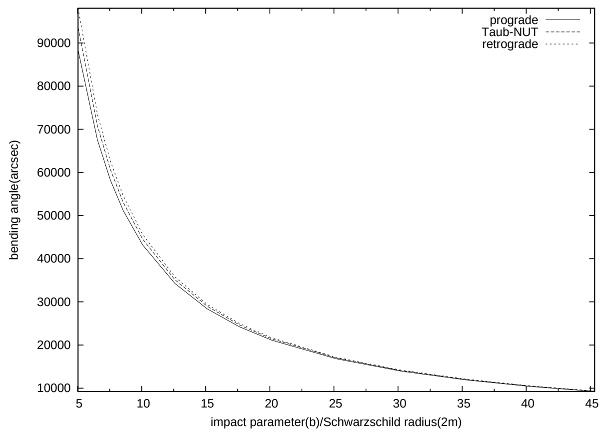

Fig. 4. *Bending angle (arcsec) as a function of impact parameter b/r_g with constant rotation ($\hat{a} = 0.5$) and the NUT factor $l = 1.413850947 \times 10^{-6}$. Geometries are explained in the caption for Fig. 1.*

Table 1. Comparison between $\alpha(\hat{a}, l)$ and $\alpha(a)$ by taking the Sun as a test case.

$(\hat{a})$	$\alpha(\hat{a}, l)$	$\alpha(a)$	$\frac{\alpha(\hat{a},l)-\alpha(\hat{a})}{\alpha(\hat{a})}$
0	1.7506562043022	1.7506555712511	3.6160805×10^{-7}
.2	1.7506554034600	1.7506547704096	3.6160778×10^{-7}
.4	1.7506544860391	1.7506538529890	3.6160770×10^{-7}
.6	1.7506534520394	1.7506516684129	3.6160739×10^{-7}
.8	1.750015680	1.75001595	1.99×10^{-7}
-.2	1.7506568885657	1.7506562555139	3.6160828×10^{-7}
-.4	1.7506574562505	1.7506568231980	3.6160856×10^{-7}
-.6	1.7506579073566	1.7506572743034	3.6160887×10^{-7}
-.8	1.7506582418840	1.7506576088300	3.6160925×10^{-7}

To understand the effect of the NUT factor more specifically, a comparison has been made between the light deflection angle as a function of both the rotation parameter and NUT factor $\alpha(\hat{a}, l)$ against the deflection angle as a function of only the rotation parameter $\alpha(\hat{a})$ using the following factor:

$$\frac{\alpha(\hat{a}, l) - \alpha(\hat{a})}{\alpha(\hat{a})}.$$

We consider the Sun as the test case and reproduced the results in Table 1. It is clear from the table that the NUT factor does have some noticeable effect on the light deflection angle which increases with the reduction of the rotation parameter. For the Sun the effect is in the order of 10^{-7} arcsec. In this study, we concentrate only on the equatorial deflection angle of light in the KTN background.

Abdujabbarov et al. (2008) reported the relation between the NUT factor and the magnetic field of a gravitating body. Our work will allow the calculation of the value of NUT factor and by using the calculation of Abdujabbarov et al. (2008), one will be able to obtain the magnetic field of any gravitating body. It is known that pulsars have a very high magnetic field. Our work can also be used to obtain all the physical parameters of pulsars, i.e the mass, rotation parameter, magnetic field.

7. CONCLUSIONS

From the above study the following may be concluded:

1. Expression for the equatorial deflection of light due to a KTN body has been calculated considering contributions from the mass and NUT factor up to fourth order terms.

2. The NUT factor has a noticeable effect on the path of the light ray. When compared with the Kerr expression for bending, we find that there are some extra terms in the expression for deflection which occurred due to the presence of the NUT factor. If the NUT factor is set to zero, the deflection angle gets reduced to that of the Kerr deflection angle. When we compare the effect of the NUT factor with that of static charge, we could see that these two parameters have opposite effect on the space-time geometry.

3. As we know, with the KTN metric even with a hypothetical body of zero mass and non-zero NUT factor, one can have some effect on the space-time curvature.

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СКРЕТАЊЕ СВЕТЛОСТИ У ЕКВАТОРИЈАЛНОЈ РАВНИ ПОД УТИЦАЈЕМ КЕР-ТАУБ-ЊУТ ТЕЛА

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Према Општој теорији релативности, маса, ротација, наелектрисање и присуство космолошке константе могу утицати на путању зрака светлости. Осим ових утицаја многи аутори нашли су и утицај гравитомагнетизма на путању зрака светлости. У овој студији дискутовали смо ефекат ротирајућег Кер-Тауб-ЊУТ тела, где је снага гравитомагнетног монопола представљена ЊУТ фактором или магнетном масом. Користимо методу нултог геодезика фотона да добијемо угао скретања зрака светлости за Кер-Тауб-ЊУТ тело у екваторијалној равни, све до члана четвртог реда. Наша студија показује да ЊУТ фактор има приметан ефекат на путању зрака светлости. За магнетизам једнак нули, израз за угао скретања се своди на Керов угао скретања. Међутим, за хипотетичко безмасено магнетно тело добили смо ненулти угао скретања.