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THE PROXIMITIES OF ASTEROIDS AND CRITICAL POINTS OF THE DISTANCE FUNCTION

S. Milisavljević

Department of Astronomy, Faculty of Mathematics, University of Belgrade, Studentski Trg 16, 11000 Belgrade, Serbia

E–mail: *slavisamilisavljevic7@yahoo.co.uk*

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SUMMARY: The proximities are important for different purposes, for example to evaluate the risk of collisions of asteroids or comets with the Solar-System planets. We describe a simple and efficient method for finding the asteroid proximities in the case of elliptical orbits with a common focus. In several examples we have compared our method with the recent excellent algebraic and polynomial solutions of Gronchi (2002, 2005).

Key words. Celestial mechanics – Minor planets, asteroids

1. INTRODUCTION

With regard that the number of minor bodies of the Solar System is very large and that the present catalogues include a couple of hundred thousands asteroids, the probability that the orbits for a couple of them fall close to each other in the phase space is very high. For this reason the problem of the smallest mutual distances, known as proximities, occupies an important place in astronomical studies. The behaviour of minor planets during a proximity offers the possibility to determine their masses and (some) other characteristics.

The first to begin the work on this problem were Gould (1858) and d'Arest (1860). Bearing in mind the relative distribution of orbits of asteroids they supposed that the proximities of these orbits could be expected in the vicinity of the relative nodes. These problems were also studied by Strömgren (1959), but also in his original way much before, by famous mathematician Gauss (1802). Nevertheless, the equations which, in the given scientific-historical circumstances, found a particular application in the solving of the proximity problem are due to the Litrow (1831). His equations can be written in the following form:

$$\begin{aligned}
&\alpha\sin(E+B) - a^2e^2\sin 2E + \alpha'\sin(E+B')\cos E_1 + \\
&\quad + \alpha''\sin(E+B'')\sin E_1 = 0, \\
&\beta\sin(E+C) - a_1^2e_1^2\sin 2E_1 + \beta'\sin(E_1+C')\cos E + \\
&\quad + \beta''\sin(E_1+C'')\sin E = 0.
\end{aligned} \tag{1}$$

In these equations a and a_1 are the semimajor axes, e and e_1 the eccentricities of the orbits, whereas the quantities α, α', α'', β, β', β'', B, B', B'', C, C' and C'' are functions of the orbital elements of the two asteroids.

This is followed by the papers of Linser (1862) and Galle (1910), and Galle determined equations yielding the minimum distance between two orbits in the following form:

$$\lambda \sin(E_1 + \Lambda) = \frac{a}{2a_1} \sin^2 \varphi \sin 2E + \alpha \sin(E + \lambda),$$
$$\lambda' \sin(E_1 + \Lambda') = \frac{a}{2a} \sin^2 \varphi_1 \sin 2E_1 + \alpha' \sin(E + \lambda''). \quad (2)$$

The quantities $\lambda,\ \lambda',\ \Lambda,\ \Lambda',\ \alpha,\ \alpha',\ A,\ A'$ can be represented by means of cumbersome and complicated expressions, obtained after many substitutions and transformations.

In the meantime, Fayet (1906) has determined by the intersections the proximities involving 800 asteroid orbits and also periodic comets and major planets. The intersection for an asteroid is a closed curve obtained as the projection of the true asteroid orbit onto the semi-plane perpendicular to the plane of ecliptic. The accuracy of Fayet's procedure attained $0^\circ.5$–1° in the longitude and ± 0.003 AU for the distance at the proximity. In a set of 320,000 intersection pairs, Fayet found six pairs for which the mutual distance of asteroids in the proximity did not exceed 0.0004 AU or $60,000$ km.

By limiting the treatment to the cases of coplanar asteroids or, in other words, to those cases where the longitude of ascending node and inclination were approximately equal, Lazović (1964, 1967) and Mišković (1974) have applied this method as well and, with some amendments, it is still in use.

Considering various aspects and possibilities for determining the proximities Lazović also developed mixed numerical-graphical methods. In one of such cases, Lazović (1974) derives equations of straight lines where the true anomaly appears as the parameter. The development and application of computers resulted in abandoning of the graphical methods but, in principle, there were still adapted forms of these methods developed by Lazović (1976, 1978).

The same author (Lazović 1970, 1971) presented the analysis and calculated the perturbations for pairs of quasicoplanar asteroids, whereas Lazović and Kuzmanoski (1974, 1976) gave some results concerning the duration of the proximities, as well as the changes of the mutual distances caused by the changes of the orbital elements. By calculating the proximities for the orbits of asteroids Ceres, Pallas, Juno, Vesta and other numbered minor planets, Lazović and Kuzmanoski (1983) obtained a proximity of only 0.0000154 AU, i.e. only 2300 km between (2) Pallas and (1193) Africa. Simovljević (1979) gave analytical expressions for perturbation effects in orbits of asteroids during a proximity.

Recently, we met different approaches to the proximity problem. Kholshevnikov and Vassiliev (1999) succeeded in simplifying the function of distance between two orbits by applying various substitutions. Gronchi (2002) determined the total, i.e. the maximum possible, number of stationary points in the distance function and gave the dependence of their number on the geometry of their orbits, also essential analytical improvements.

By using the true anomalies, Lazović (1974) starts from the condition that the heliocentric position vectors $\vec{r}_1$ and $\vec{r}_2$ for the first and second elliptical orbits should be expressed via the true anomalies $(\upsilon_1,\ \upsilon_2)$ in the form:

$$\vec{r}_1 = r_1 \cos \upsilon_1 \vec{P}_1 + r_1 \sin \upsilon_1 \vec{Q}_1,$$
$$\vec{r}_2 = r_2 \cos \upsilon_2 \vec{P}_2 + r_2 \sin \upsilon_2 \vec{Q}_2, \quad (3)$$

and that the relative position vector is equal to $\vec{\rho} = \vec{r}_1 - \vec{r}_2$. The square of the distance will also be a function of the true anomalies and the conditional equations for the existence of extremum will have the following form:

$$\frac{\partial \rho^2}{\partial \upsilon_1} = 0, \quad \frac{\partial \rho^2}{\partial \upsilon_2} = 0. \quad (4)$$

After solving and transforming he obtains expressions of the form:

$$f(\upsilon_1, \upsilon_2) = 0, \quad g(\upsilon_1, \upsilon_2) = 0, \quad (5)$$

i.e. a system of transcendent equations expressed via the true anomalies, which he solves by applying successive approximations until the following values are found:

$$\upsilon_{1n} = \upsilon_{1(n-1)} + \Delta\upsilon_{1(n-1)},$$
$$\upsilon_{2n} = \upsilon_{2(n-1)} + \Delta\upsilon_{2(n-1)}, \quad (6)$$

satisfying the initial system of equations to a necessary accuracy level.

2. FORMULATION OF THE PROBLEM

With regard that the orbits of asteroids are elliptical, speaking strictly mathematically, the problem consist in calculating all the possible minimum spatial distances between two ellipses with a common focus.

After applying some substitutions, system (4) becomes expressed through algebraic forms, and this results in simplifying the distance function (Kholshevnikov and Vassiliev 1999). Gronchi (2002) gives the distance function as the square of the distance, but in a rectangular coordinate system. He uses Bernstein's (1975) theorem which concerns the solution intersection of two polynomials with two variables in the complex field of numbers and by applying Minkowski's sum he calculates the total number of solutions for the system. Based on this he determines the number of stationary points for the distance function depending on whether the orbits are circular or elliptical. This situation can be better appreciated from Table 1 where the eccentricities (e and e') of the first and second asteroid orbits are used.

Table 1.

eccentricity of the first orbit	eccentricity of the second orbit	number of stationary points
$e \neq 0$	$e' \neq 0$	12
$e \neq 0$	$e' = 0$	10
$e = 0$	$e' \neq 0$	10
$e = 0$	$e' = 0$	8

The first question here is if there are more characteristic positions (apart from the vicinity of relative nodes) where proximities can be expected. Then, one can ask whether this is possible to conclude based on a mere inspection of the orbital elements.

With regard that a proximity means a minimum of the mutual distance of two elliptical orbits and knowing that at the proximity both vectorial Eqs. (3) are satisfied simultaneously, another question arises: what about the values which also satisfy both vectorial equations, but are not proximities (saddle points and maxima), and how large can be their total number? Knowing that all these values (proximities, saddle points and maxima) are solutions of the transcendent Eqs. (5) the answers would surely explain and define the proximity problem better. A more complete analysis concerning the possible number of all extremal values of the distance function between two elliptical orbits can be found, as already said, in the papers by Kholshevnikov and Vassiliev (1999) and Gronchi (2002, 2005). Their analyses confirm the earlier hypotheses that there can be four minima, or proximities, in the distance between two elliptical orbits. They also give the number and positions for the other extremal values (stationary points) satisfying the initial system of Eqs. (5).

3. EXISTENCE OF PROXIMITIES AND CRITICAL POINTS OF THE DISTANCE

When we speak about the particular conditions which must be fulfilled in order to have a proximity between two elliptical orbits of minor planets, we should point out the following cases:

a) There is always at least one proximity. This is clear because of the well known property of two closed curves in space that at least one minimum distance must exist (Fig. 1).

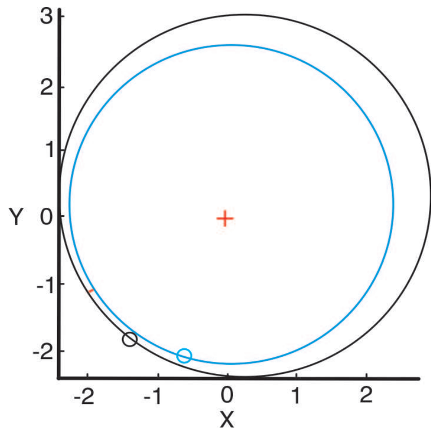


Fig. 1. *Projection of the asteroids' orbits (4-638) on the plane of the first orbit one proximity example.*

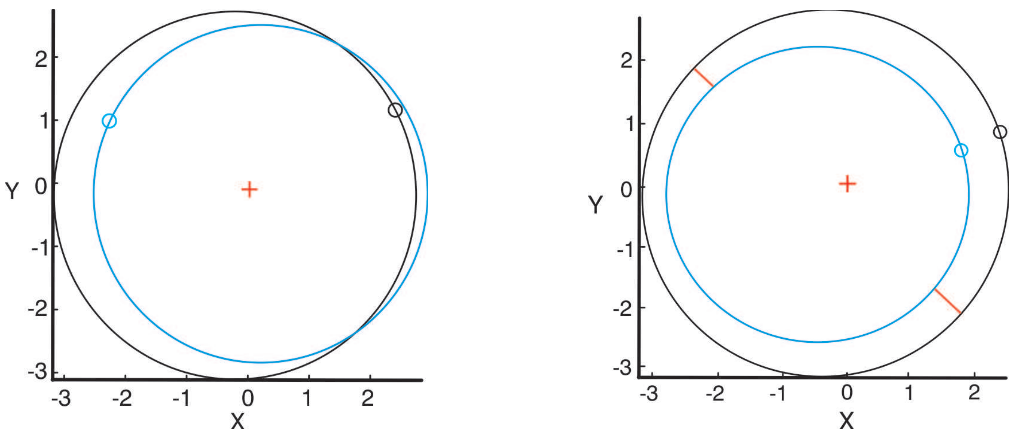
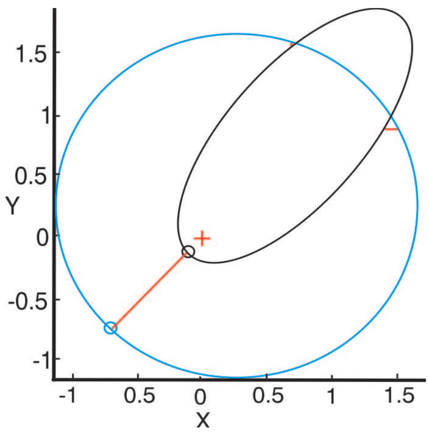


Fig. 2.a/b *Projection of the asteroid orbits (1-3468), left, and (6-16), right, on the plane of the first orbit; two proximities example.*



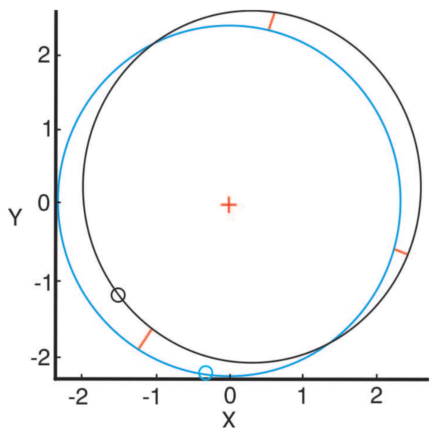


Fig. 3.a/b *Projection of the asteroids orbits (1943-3200), left, and (287-486), right, on the plane of the first orbit; three proximities example.*

b) Two proximities can exist and this is, practically, the most frequent case. They are usually located in the vicinity of the relative nodes. With regard to the relatively low mutual inclination in all the examples given here the projections onto the XY plane offer a sufficiently realistic picture of the true situation. Nevertheless, we introduce here the term projection intersection because these points need not always be in the vicinity of the relative nodes. For the case of two proximities it is enough to have an inclination between the orbital planes or viewing the projections onto the XY plane to have impression of their intersecting at two points. Figs. 2a - 2b in principle represent all the cases which are possible in reality. In Fig. 2a, the projection of the orbits for a pair of minor planets (1-3468) onto the XY plane is given whereas in Fig. 2b one finds the corresponding projection for another pair (6-16).

c) The case with three proximities is possible, but it takes place much more rarely when the orbits of asteroids are in question. The distribution of proximities is, as a rule, such that two of them are in the vicinity of the projection intersections of the orbits (more rarely in the vicinity of nodes), whereas the third one is always almost symmetrically located on the opposite side. The conditions for existence of such a case usually follow from the special positions of the previous case. This can be clearly seen from Fig. 3a for the minor-planet pair (1943-3200) and from Fig. 3b for the pair (287-486).

d) Four proximities are also possible, but in reality it is very difficult to find them. The example from Fig. 4 is a simulation of Gronchi's (2002) model and it clearly demonstrates that the case with four proximities is possible.

With regard to a very high eccentricity and also to an extremely high relative inclination (more than 80°) in this example, as well as to other examinations with similar values of ellipse elements, one is inclined to think that cases with four proximities are possible only for the pairs with characteristics similar to the simulated ones.

The fact is that the total number of proximities and maximum distances is always equal to the number of saddle points. All these values are solutions of the system of transcendent equations, i.e. stationary points of the distance function. With regard that the maximum number of intersecting points between two ellipses in the plane with a common focus is equal to 2 (Milisavljević 2002), the following question arises: What cause, typical for the distances between two ellipses in the same plane with one common focus, could in the case of a minor change of the inclination, result in three proximities?

Such a case would be when we have two common points and a third one where the two orbits are almost in contact as shown in Fig. 5. The third characteristic position (M_3) can be additional minimum distance between two ellipses provided that $SP_1 = SP_2$. This distance cannot be equal to zero, i.e. it will not become a real contact.

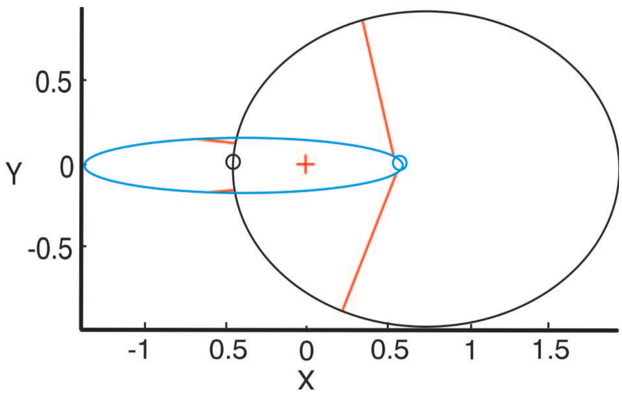


Fig. 4. *Projection of the fictious asteroid orbits (Gronchi 2002) on the plane of the first orbit; four proximities example.*

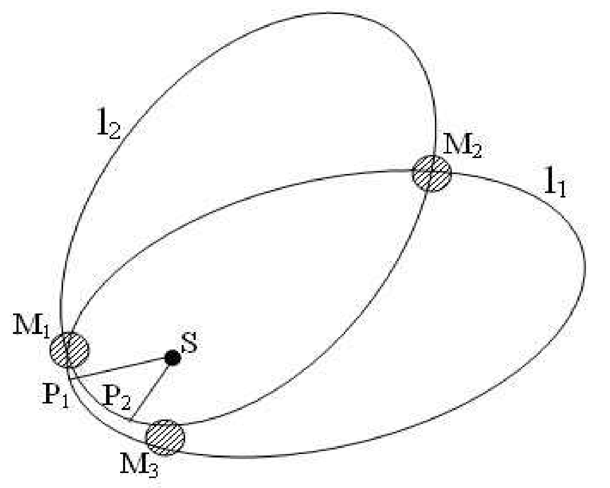


Fig. 5. *Characteristic areas of proximity in the case of 3 proximities.*

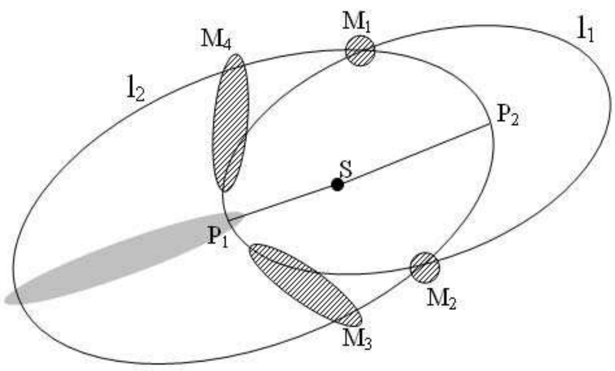


Fig. 6. *Characteristic areas of proximity in the case of 4 proximities.*

When the given conditions are fulfilled, and the situation is like that in Fig. 5 where, as a rule, the characteristic place of "contact" is near the perihelion for the rotating ellipse, we obtain a model which, in the stereometric case for non-zero inclination of the ellipses, has three proximities. Two of them are in the vicinity of the projection intersections of the orbits M_1 and M_2, whereas the third one takes place near the characteristic position M_3.

The case with four proximities could be viewed as a special case of the previous one (see Fig. 4). The two ellipses now have a position with two common points, but the perihelion positions differ by about 180° (Fig. 6).

This is not the case of the two proximities from Fig. 2a, but from this planimetric interpretation concerning the stereometric case, for a sufficiently high inclination difference of the planes in which the ellipses are, the presence of two more proximities within the zones M_3 and M_4 results. These proximities are located almost symmetrically with respect to the direction of the maximum distance between the ellipses. There are two reasons for their existence. The first is the position of the ellipses resembling "the links of a chain" (the perihelia are on the opposite sides) so that the position M_3 from Fig. 5, conditionally called "the contact point", does not exist here. On the contrary, bearing in mind that the position M_3 was in the perihelia direction, under the conditions of this mutual positioning of the ellipses, it is simply lost and corresponds to one of the largest distances (shaded part in Fig. 6). Thus we have a situation in which we remain with two proximities only, as already presented in Fig. 2a.

However, the gradual enlarging of the relative inclination between the planes of the ellipses leads to the appearance, at first, of the third proximity and then, around "the critical angle" (in the given example its value is 79°-81°), also of the fourth proximity. They are both just in the zones M_3 and M_4. It is clear that the sufficiently high inclination between the two ellipses, where we have four proximities, will not be the same for all the examples of this type, but it can be said that just this is the reason why four proximities exist.

All the other possible positions (regardless of values of orbital elements, especially of that of the relative inclination) can yield nothing which would be substantially different from the examples presented here. The last case with four proximities is the most complicated one. Together with the two maximum distances existing here and the six saddle points it has a total of 12 stationary points. This is also the maximal number of solutions to this problem which has been explicitely conjectured by (Gronchi 2002) In the same paper there is a proof that they are at most 16 in the general case (12 if one orbit is circular).

4. DESCRIPTION OF OUR ALGORITHM

The procedure for finding all possible proximities, which will be presented here, is based on an idea of Simovljević (1977) which has not been published yet.

The idea is to construct from an arbitrary point of an orbit (though it is completely unimportant from which orbit it is started) a perpendicular line to another orbit and then to construct, from the obtained point on the second orbit, the perpendicular line to the first orbit. The procedure should be continued until the proximities are found, i.e. the two perpendicular lines coincide with each other.

According to this idea one should use the fact that the two position vectors $\vec{r_1}$ and $\vec{r_2}$ are known at every point of the orbit which yields the relative position vector $\vec{\rho}$. Its modulus at the point of the proximity determines the distance at the proximity itself.

Here we do not want to introduce any approximations. Instead, we establish the convergence of the relative position vector $\vec{\rho}$ provided that the limiting case of its convergence is always at one of the possible proximities. In this way, by calculating the last value of the relative position vector, we can also calculate the proximity. We shall have as many proximities as there are such convergence while we shift along one of the orbits.

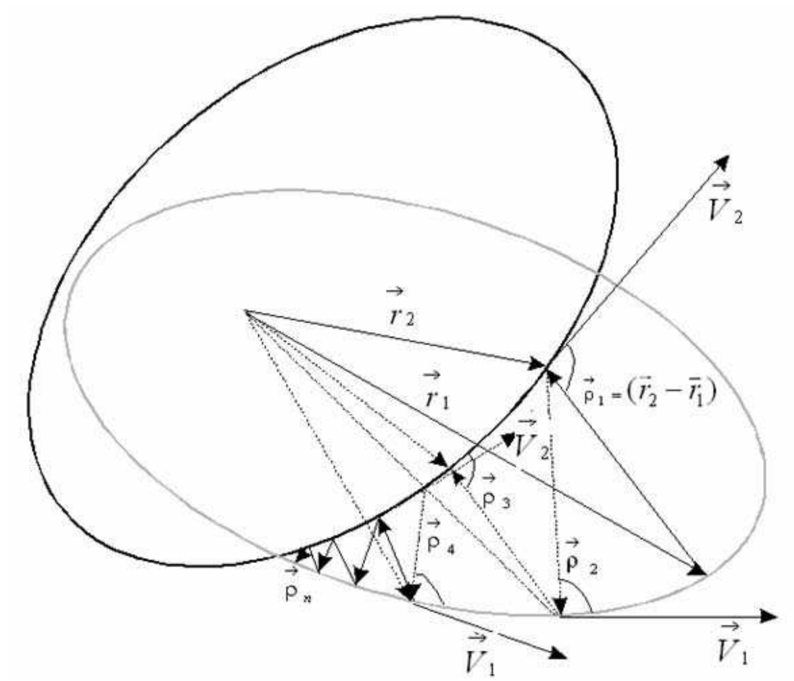


Fig. 8. *Geometrical illustration of the proximity problem solving.*

Let us denote $\vec{\rho}_1, \vec{\rho}_2, \vec{\rho}_3, ..., \vec{\rho}_n$ the corresponding vectors after the i-th calculation, i=1,...,n

We are starting, as a rule, from the perihelion ($v_1 = E_1 = 0$) but we can also construct the relative position vector $\vec{\rho}$ from an arbitrary point of the first orbit. Fig. 7 illustrates such a procedure but for the sake of better appearance and understanding the relative position vector starts from aphelion. It doesn't change anything in process of calculation as we demonstrated before. This vector must be perpendicular to the tangent at the corresponding point of the orbit of the other asteroid. One of the two equations known for the proximity condition is the vector equation for the given procedure, i.e. the following equation:

$$(\vec{r}_2 - \vec{r}_1) \cdot \frac{d\vec{r}_2}{dt} = 0. \tag{7}$$

Since E_1 has already been chosen (usually it is initially $E_1 = 0$), the unknown quantity in this vector equation is E_2 only. It can be calculated by using the data concerning a particular case. After calculating E_2 we have a point on the other orbit which is the end point of $\vec{\rho}$, the relative position vector. Then the procedure is repeated, but this time the origin of $\vec{\rho}$ is at the point of the second orbit for which the calculated value of eccentric anomaly is E_2.

The end point of the relative position vector $\vec{\rho}$ for the given point of the first orbit is perpendicular to the tangent at this point. Thus:

$$(\vec{r}_1 - \vec{r}_2) \cdot \frac{d\vec{r}_1}{dt} = 0 \tag{8}$$

and now the unknown quantity is E_1. All other quantities are known, including E_2, as well, obtained from the previous calculation. It is seen that the procedure is repeated until the "departing" and "arriving" vectors have equal moduli and directions or, in other words, both equations are satisfied to a given accuracy (established by the user at the beginning of the procedure). The two last values for E_1 and E_2 in the calculation process are the values for eccentric anomalies for which the minimum distance is attained, the last calculated value for the modulus of $\vec{\rho}$. According to the present geometric procedure one assumes that the position vectors for the minor planets are expressed in terms of eccentric anomalies E_1 and E_2, i.e.

$$\begin{aligned} \vec{r_1} &= a_1(\cos E_1 - e_1)\vec{P_1} + b_1 \sin E_1 \vec{Q_1}, \\ \vec{r_2} &= a_2(\cos E_2 - e_2)\vec{P_2} + b_2 \sin E_2 \vec{Q_2}. \end{aligned} \tag{9}$$

Bearing in mind that $\dfrac{d\vec{r}}{dt} = \vec{V}$ and using the relation $\dfrac{d\vec{r}}{dt} = \dfrac{d\vec{r}}{dE}\dfrac{dE}{dt}$, Eqs. (8) and (7) can be written as:

$$\begin{aligned} (\vec{r}_1 - \vec{r}_2) \cdot \frac{d\vec{r}_1}{dE_1} &= 0, \\ (\vec{r}_2 - \vec{r}_1) \cdot \frac{d\vec{r}_2}{dE_2} &= 0. \end{aligned} \tag{10}$$

In what follows the system (10) is solved step by step as we described above.

Only one of the four possible roots is of interest to us. It is defined as the smallest value among all possible differences of the moduli of vectors $\vec{r}_2$ and $\vec{r}_1$. We can find the corresponding eccentric anomalies and also the moduli of the position vectors. By subtracting the modulus of $\vec{r_1}$ from each modulus of $\vec{r_2}$ we obtain four values for the modulus of $\vec{\rho}$. Their comparison yields the smallest among them. The corresponding eccentric anomaly is the solution we were looking for.

5. COMPARATIVE UNCERTAINTIES AND ADVANTAGES OF OUR METHOD

The other solutions of our vector equations – saddle points and maximum distances – cannot be directly obtained by using this procedure. The reason is that the relative position vector $\vec{\rho}$ is always perpendicular to the tangent at the orbit point where it ends up so that, consequently, it always converges to the minimum distances between two elliptical orbits. Vice versa, if it were perpendicular to the tangent at the orbit point of its origin, in some cases we could have a convergence to the saddle points and maximum distance between two elliptical orbits. However, while the procedure for determining the minimum distance or proximity calculates (regardless of the relative position and sizes of the elliptical orbits) all proximities, the converse procedure doesn't guarantee determining of all saddle points and maximum distances, but this depends on the particular case.

Due to this reason, the converse procedure is not used here. Instead, the positions of the saddle points and of the maximum are found directly from the plots such as the plot of the distance function. If more precise values are needed, it is possible to use Lazović's (1993) approximative procedure.

The procedure for proximity determination presented here guarantees that both orbits are searched completely so that there is no risk for any possible proximity not taking place at an expected position (usually in the vicinity of relative nodes) to evade the detection.

On the basis of geometric presentation in the present work an analytical procedure is derived and the corresponding software (algorithm and programme) is developed.

In the case of the earlier methods, the two vector equations:

$$\begin{aligned}(\vec{r}_1 - \vec{r}_2) \cdot \vec{V}_1 = 0,\\ (\vec{r}_2 - \vec{r}_1) \cdot \vec{V}_2 = 0,\end{aligned} \tag{11}$$

defining the problem, have been treated together only, i.e. as a system. With regard that Eqs. (11) are transcendent, they have been solved largely by using successive approximations, no matter whether the variables were true or eccentric anomalies.

The main characteristic of our program is that it always starts from $E_1 = 0$ for every pair of asteroids, but it works for every point on orbit as we can see on Fig. 7. Different input data (E_1=5°, E_2=10°, ..., E_n=n·5°) is first find an area where there is proximity, and then access to accurate calculation. The reason for this approach is not to miss any minimum distance and saddle in the area covered.

If $|\vec{\rho}_1| > |\vec{\rho}_2|$ (i.e. in general case $|\vec{\rho}_{n-1}| > |\vec{\rho}_n|$), there is no need to increase the following input value for E by 5 degrees because the proximity is ahead.

If $|\vec{\rho}_1| < |\vec{\rho}_2|$ (i.e. in general case $|\vec{\rho}_{n-1}| < |\vec{\rho}_n|$), then the following input value for E increases by 5 degrees. That means that the saddle is ahead and we repeat the procedure until we locate region which contains proximity.

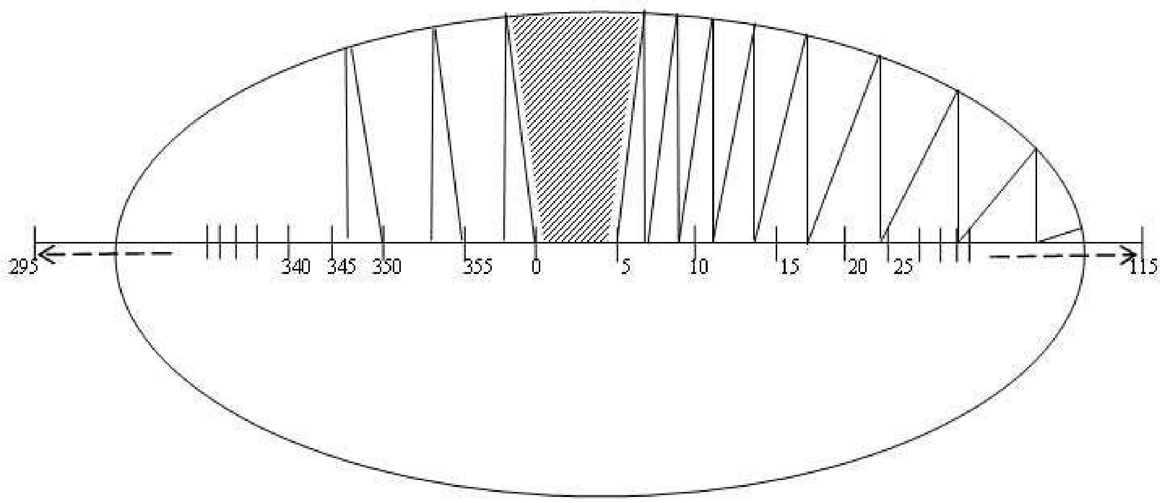


Fig. 8. *Geometrical illustration of the saddle problem solving, during the proximity calculation.*

Table 2.

Group	Type	Number of minima	Number of maxima	Number of saddle points	Total number of stationary points
I	A	1	1	2	4
	B	1	2	3	6
II	A	2	1	3	6
	B	2	2	4	8
III	A	3	1	4	8
	B	3	2	5	10
IV	B	4	2	6	12

As a matter of principle we search the whole orbit in order not to miss any possible proximity.

Due to the continues alteration of the minimum distances and saddle points we shift the departing vector always to a new initial position until it crosses the neighboring saddle point and ceases to get back to proximity calculated earlier (Fig. 8). If the first elliptical orbit were situated in the xy plane oriented edge on, we will see it as a horizontal line. The other one would in such a case be represented with the its elliptical projection as shown Fig. 8 (in this particular case the major axes of two orbits are aligned for better representation but this in general is not the case). The plot thus corresponds to the projection of these two elliptical orbits onto the plane XZ or YZ depending on the other parameters.

It is to be said here that, though the values for ρ at some saddles can even be smaller than the minima, i.e. than the proximities, this fact has no serious influence on the method and functionating of the programme. The reason is that it can never take place between "neighboring minima", but at other characteristic positions, so that the alternation between narrower and wider areas in the band, where the procedure takes place, is always present and, consequently, no disturbances are possible during the search and calculations.

6. APPLYING THE METHOD TO THE SET OF ASTEROIDS

By applying of the procedure presented above we have examined the proximities for about 600,000 pairs of asteroids. Here we present the results for selected examples of every type (Table 2). The other results are given in Appendix.

Note that in no case can we have a pair of asteroids which would not belong to any of the types from these four groups.

On the basis of this classification we can conclude that the two transcendental Eqs. (3), describing the proximity problem, can have 4, 6, 8, 10 or 12 solutions. Depending on the values of the corresponding partial derivatives, these solutions can be minima, maxima or saddle points.

Analyzing all the obtained results, the following comment becomes possible:

a) In the first group of pairs for both types one finds very close inclination values so that the quasi-coplanarity condition is here most present. Nevertheless, it does not necessarily guarantee close proximities. The reason is that no projection intersection of orbits exists here. They are always inside each other and, consequently, a small inclination difference results in no close proximities. The positions and values of the saddle points, no matter whether there are two or three of them, are as expected because they fulfil the condition of being longer the minimum and being smaller than the maximum (which need not always be the case). For type A the maximum distances are usually at about 180^0 with respect to the proximity position, whereas in the case of type B they can have various positions.

b) The second group of pairs occurs in reality most frequently, especially type A. Due to this the criteria for existence of asteroids pairs of this type are the weakest. The positions of the proximities are most frequently in the vicinity of the nodal line, whereas the distribution of the saddle points and maxima is quite irregular. In this group very close perihelion positions are possible for the case of type B.

c) The third asteroid group is the most rarely met and they are hardly found among quasi-coplanar asteroids. One of the three proximities is almost always very close to one of the perihelia, whereas the other two are within a somewhat wider neighborhood of the nodal line. Their values are sufficiently close to each other and the peaks in the plot of the function $1/\rho$ are not so prominent. In the case of type B from this group the situation is very specific and complex so that the finding requires all conditions to be strictly fulfilled. The projection of the pair (1943-3200) onto the XY plane shows that it looks just like as we predicted theoretically in considering possible positions of orbits with three proximities. Though the values of the proximities are high, they take place at the expected positions or sufficiently close to them. A symmetry concerning the major axes of both orbits (they almost coincide), which exists in this example, causes two maxima and even as much as saddle points which, after adding three minima, means a total of ten solutions, i.e. pairs of values for E_1 and E_2 satisfying both initial equations. What is curious in this example is the fact that one of the values for the relative position vector at the saddle point is smaller than the corresponding proximity value. If we had in reality a case belonging to type B from this group, but with all the three proximities sufficiently small, then a simultaneous passage of both minor planets through such a saddle point might be treated as a close encounter.

d) The fourth group and the model of two elliptical orbits with four proximities, two maxima and six saddle points, resembles the example from the preceding group (1943-3200). The difference concerns the opposite position of the perihelia (one of them is shifted by about 180° with respect to the other one) and, of course, the much higher relative inclination of the orbital planes. The positions of the proximities are at places where they are expected to be located and their values are mutually close enough. Here the modula of the relative position vector at the saddle point can also be smaller than a proximity, whereas the position of one of the maximum distances is, due to the opposite perihelion orientation, always in the middle of the diagram.

7. CONCLUSIONS AND FUTURE WORK

Thanks to the analysis carried out here it became possible to detect the existence of asteroids pairs where three proximities can take place. It has been also shown that the case with four proximities is possible, as it has been supposed, but that such a case is at present very difficult to find among real asteroids (if such exists at all with regard to their relative inclinations) due to the exceptionally rigorous conditions concerning its existence. The presented model with four proximities was obtained by Gronchi (2002) by applying the method of random samples, i.e. after many simulations and trials with various values of elliptical elements.

The fact that the modulus of the relative position vector $\vec{\rho}$ at some saddle points can be smaller than at a proximity can also be pointed out, especially if the proximities are very light. Such cases surely deserve attention and can be among the topics of the future work concerning the proximity problems.

Finally it should be said that the main aim of the present work was to finally solve the dilemma: how many extreme distances are really there between two asteroids on their orbits and which kind of relationship can they have? Bearing in mind the importance of studying close encounters because of the mass determination, as well as the fact that the possibility of a close encounter concerning the Earth is always present, such and similar analysis contribute to a better understanding of asteroids.

REFERENCES

Fayet, G.: 1949, Contribution à l'ètude des proximitèn d'orbites dans le système solaire, Ann. Bur. des Long., t. XII, Centre National de la Rech, Scient., Paris.

Gronchi, G. F.: 2002, On the stationary points of the squared distance between two ellipses with a common focus, Dept. of Mathematics University of Pisa.

Gronchi, G. F.: 2005, An algebraic method to compute critical points of the distance function between two Keplerian orbits. Dept. of Mathematics University of Pisa.

Grunert, J. A.: 1854, Über die Proximitäten der Bahnen der Planeten und Cometen, Sitzb. d. math. – naturw. Cl. der k. Ak. d. Wiss., Wien.

Kholshevnikov, K. and Vassiiliev, N.: 1999, On the distance function between two Keplerian elliptic orbits, St. Petersburg. University St.Petersburg Branch of the Steklov Institute of Mathematics, Russian Academy of Sciences.

Kuzmanoski, M. and Knežević, Z.: 1993, *Icarus*, **103**, 93.

Lazović, J. P.: 1964, Važnije osobenosti u kretanju kvazikomplanarnih asteroida. Doktorska disertacija, Beograd.

Lazović, J. P.: 1967, *Bull. ITA*, **1**, 57.

Lazović, J. P.: 1974, *Publ. Dept. Astron. Belgrade*, **5**, 85.

Lazović, J. P.: 1993, *Publ. Astron. Obs. Belgrade*, **44**, 11.

Lazović, J. P. and Kuzmanoski, M.: 1983, *Public. Dept. Astron. Belgrade*, **12**, 11.

Lazović, J. P. and Kuzmanoski, M.: 1985, *Public. Dept. Astron. Belgrade*, **13**, 13.

Littrow, K. V.: 1854, Bahnnähenywischen den periodischen Gestirnen des Sonnensystems, Sitzb. d. Math. – Naturw. Cl. der k. Ak. d. Wiss., Wien.

Mišković, V. V.: 1974, *Glas srpske akademije nauka i umetnosti (Odeljenje prirodno-matematičkih nauka)*, **37**, 31.

Simovljević, J. L.: 1977, *Glas srpske akademije nauka i umetnosti (Odeljenje prirodno-matematičkih nauka)*, **41**, 65.

Simovljević, J. L.: 1979, *Acta Astron.*, **29**, 445.

APPENDIX

In this appendix we present some of our numerical results in the following three tables.

Table A1. Comparative results for given pairs of minor planets. The first part contains examples with one and two proximities. The second part contains examples with three and four proximities.

GOUP TIP	ASTEROID PAIR		NUM. MIN.	NUM. MAX.	E1				E2				ρ			
					1	**2**	**3**	**4**	**1**	**2**	**3**	**4**	**1**	**2**	**3**	**4**
I A	**4**	**638**	1	1	318.967433				340.249584				0.11881608			
	5	**933**	1	1	312.706654				297.302111				0.047196291			
	5	**7015**	1	1	299.722771				272.883825				0.024417198			
	6	**9038**	1	1	234.851250				253.975712				0.040672117			
	6	**17367**	1	1	112.398698				90.992470				0.0065865653			
	8	**16665**	1	1	224.844783				251.317369				0.044100568			
I B	**6**	**5651**	1	2	236.500346				243.517153				0,67499933			
II A	**1**	**3468**	2	1	149.074598	266.231525			283.877566	26.989619			0.010086978	0.015192299		
	1	**6358**	2	1	66.118347	291.341853			261.103638	100.158154			0.0098925052	0.0081538673		
	3	**3883**	2	1	111.812402	303.266840			194.456494	357.732910			0.007961969	0.0077208055		
	3	**4117**	2	1	119.462212	293.389098			250.107399	23.822538			0.017956696	0.0047575898		
	3	**4502**	2	1	67.042155	253.598784			356.551117	151.919450			0.005247323	0.011706341		
	3	**5001**	2	1	113.484941	306.275822			232.045497	26.555078			0.0079492381	0.0002694883		
	3	**5007**	2	1	120.442905	311.448417			227.540475	22.071432			0.016495841	0.0091743864		
II B	**2**	**13**	2	2	55.633571	261.555109			58.273631	247.507784			0.048952648	0.21530592		
	6	**16**	2	2	116.026033	302.073699			116.159223	299.344475			0.47502977	0.56712944		
	6	**18**	2	2	87.711718	288.546055			87.253306	288.575483			0.13666059	0.13546653		
	8	**19**	2	2	45.361053	257.466472			47.182367	261.378162			0.22475493	0.23055124		
	16	**18**	2	2	146.609615	291.312721			146.531778	294.090475			0.5675111	0.70773182		

Table A1. (continued)

GROUP TIP	ASTEROID PAIR		NUM. MIN.	NUM. MAX.	E1				E2				ρ			
				1	1	2	3	4	1	2	3	4	1	2	3	4
III A	5	29	3	1	121.109388	273.260901	338.168369		213.185130	343.388641	50.894535		0.51315999	0.44341348	0.45460981	
	33	39	3	1	19.327834	82.188329	254.932808		6.941324	77.469594	221.642821		0.55819276	0.54373861	0.57805759	
	45	387	3	1	35.244275	207.488698	300.603308		350.002033	147.350762	259.631433		0.48365951	0.54683653	0.52228889	
	57	439	3	1	86.905478	235.416276	353.895579		66.650556	207.589915	331.088189		0.2113557	0.23089925	0.19149363	
	58	387	3	1	90.135240	211.404643	336.337156		0.596015	104.624437	257.048828		0.62333288	0.68247902	0.59913665	
	73	216	3	1	96.496953	200.075592	336.044609		108.828459	236.248118	359.524241		0.70217504	0.71816082	0.47701421	
	74	200	3	1	85.079532	208.029051	356.290882		54.494041	160.912391	321.907103		0.41312331	0.50917599	0.36440214	
	90	294	3	1	166.839776	255.497150	353.227923		150.199275	248.341303	344.350031		0.29747009	0.26160957	0.31167185	
	177	205	3	1	4.023781	128.863248	231.609991		5.172109	138.426909	222.784858		0.55167579	0.52809531	0.51026427	
	195	289	3	1	17.382783	85.670748	257.708693		123.090712	208.885295	8.711346		0.62192967	0.64808489	0.63963437	
	215	216	3	1	34.540416	197.834012	285.106901		346.603279	133.949099	244.115517		0.61169318	0.62749076	0.68965614	
	287	486	3	1	82.955763	175.228201	336.229042		118.568066	224.566002	15.644404		0.3378866	0.3317445	0.34319174	
	308	410	3	1	104.657248	231.837321	280.945388		118.226247	267.305685	312.384668		0.42093247	0.55517338	0.55665726	
	324	353	3	1	1.741296	46.326779	202.057802		325.184125	9.086711	131.154659		0.50022138	0.4986536	0.92224752	
	378	415	3	1	34.376689	164.810410	287.380643		2.612966	115.657336	262.160593		0.55485577	0.64875647	0.42682285	
	442	463	3	1	57.671837	108.925052	254.890643		287.324885	332.541002	91.074818		0.61011463	0.61119662	0.6483809	
	85	1021	3	1	71.046932	217.929628	348.818275		4.586064	119.872994	287.466257		0.65426741	1.0305813	0.7279546	
III B	1943	3200	3	2	0.937111	160.783606	207.629896		359.954690	114.935624	247.205530		0.92727839	0.40194025	0.22666053	
IV	MP1	MP2	4	2	8.230714	96.894151	255.144312	339.552785	291.754111	355.364496	11.063867	67.551575	0.86963111	0.83240761	0.95536426	0.94555766

Table A2. Results for minor planet pair 1943-3200.

E1	E2	ρ	
0.93711	359.95469	0.927278	**Minimum**
160.78361	114.93562	0.401940	**Minimum**
207.62990	247.20553	0.226661	**Minimum**
3.49866	180.48346	3.454164	**Maximum**
180.11995	359.39036	1.933373	**Maximum**
183.05929	177.78636	0.722612	**Saddle**
119.45135	316.76053	1.830068	**Saddle**
64.37153	39.75393	0.983832	**Saddle**
246.76346	45.15348	1.803981	**Saddle**
308.07886	326.30679	0.963051	**Saddle**

Table A3. Results of the simulated model for minor planets MP1 and MP2.

E1	E2	ρ	
1.135	343.459	1.0527	**Maximum**
5.282	204.023	0.9703	**Saddle**
355.364	96.894	0.8324	**Minimum**
11.064	255.144	0.9554	**Minimum**
346.325	323.537	1.0487	**Saddle**
291.754	8.231	0.8696	**Minimum**
34.551	302.158	0.9757	**Saddle**
321.611	40.033	0.8982	**Saddle**
67.552	339.553	0.9456	**Minimum**
30.399	19.201	1.0308	**Saddle**
181.953	173.950	3.3464	**Maximum**
177.495	352.694	1.3528	**Saddle**

ПРОКСИМИТЕТИ АСТЕРОИДА И КРИТИЧНЕ ТАЧКЕ ФУНКЦИЈЕ РАСТОЈАЊА

S. Milisavljević

Department of Astronomy, Fauculty of Mathematics, University of Belgrade
Studentski Trg 16, 11000 Belgrade, Serbia

E–mail: *slavisamilisavljevic7@yahoo.co.uk*

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Оригинални научни рад

Проксимитети су важни за различите сврхе, на пример за процену ризика судара астероида или комета са планетама Сунчевог система. Описан је једноставан и ефикасан метод за тражење астероидских проксимитета у случају елиптичних путања са заједничком жижом. У неколико примера изложени метод је поређен са најновијим и изванредним алгебарским и полиномским решењима (Gronchi 2002, 2005).