Serb. Astron. J. № 167 (2003), 27 - 34

UDC 52–645
Original scientific paper

A FORTH-AND-BACK IMPLICIT Λ ITERATION IN THE SOLUTION OF RADIATIVE TRANSFER IN SPHERICAL MEDIA

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(Received: October 10, 2003; Accepted: October 23, 2003)

SUMMARY: Forth-and-back implicit Λ-iteration has been developed to solve radiative transfer (RT) problems with plane-parallel geometry in which there is a coupling of all RT equations by the scattering term included in the source function (Atanacković-Vukmanović, Crivellari and Simonneau 1997). Owing to the implicit representation of the source function in the computation of the mean intensities within a forth-and-back sequential treatment of the two intensities propagating in opposite directions, implicit Λ-iteration (ILI) appears to be a very efficient method in the solution of linear as well as non-linear transfer problems. In this paper ILI method is generalized and applied to radiative transfer problems with spherical symmetry. The results for the monochromatic radiative transfer in a spherical atmosphere are presented and compared to those of other authors obtained by the other methods.

Key words. Methods: numerical - Radiative transfer - Stars: atmospheres

1. INTRODUCTION

Radiative transfer (RT) is a physical phenomenon essential to many astrophysical problems and is one of the most difficult to deal with. The main difficulty with the radiative transfer problems comes from the non-local and, in general, non-linear coupling of the radiation field and the state of the gas. The specific intensity of the radiation field at each point of a medium depends, via radiative transfer process, on the state of the gas over a wide range of distant points, whereas the state of the gas depends, through the interactions with the radiation field, upon the radiation field intensity itself. Mathematically, this coupling is accounted for by the simultaneous solution of the corresponding equations, one describing the dependence of the mean intensity on the source function $J = \Lambda[S]$ through RT process (by means of the so-called Λ-operator), and the other defining the source function in terms of the mean intensity of the radiation field $S = S(J)$. If the coupling is linear, i.e. if the latter dependence has an explicit form, RT problem can be solved by using either direct or iterative methods. However, in more general non-linear cases, some kind of iterative procedure is required.

The most straightforward iterative procedure, the so–called Λ iteration, solves the two coupled equations in turn. The solution of the problem can be represented by the following sequence: $S^o(\tau) \rightarrow I(\tau,\mu) \rightarrow J(\tau) \rightarrow S^n(\tau)$. With the given source function $S^o(\tau)$ (a trial solution to start the procedure or the one obtained from the previous iteration step), the solution of the RT equation yields specific intensities of the radiation field $I(\tau,\mu)$ and, hence, the mean intensity $J(\tau)$ which is used to update the value of the source function $S^n(\tau)$. However, in most cases of interest the rate of convergence of this simple procedure is extremely slow.

The first method developed to solve the problem in a fully self-consistent manner was that of complete linearization introduced by Auer and Mihalas (1969). It was widely used during 1970s. Although, from the conceptual point of view this method is entirely different from the Λ iteration, the two methods are akin in the sense that both use exact (full) RT operator. The method of complete linearization necessarily implies the storage and inversion of matrices whose dimensions (being equal to the number of discrete ordinates required for a good description of the radiation field) are extremely high. Therefore, being very time and memory consuming, it was in practice restricted to simplified atomic models and geometries. Moreover, due to the cumbersome matrix structure it may suffer from numerical instabilities.

Due to their simplicity and a smaller error because of a smaller number of numerical operations, iterative methods became again more widespread. A class of the so-called accelerated Lambda iteration (ALI) methods is developed, based on the idea of Cannon's (1973a,b) operator perturbation technique. An approximate (Λ^*) operator is introduced instead of the full (Λ) one, whereas a small error made by this approximation is computed either iteratively or by the use of perturbation procedure. In other words, some approximation is used to simplify the detailed description of the RT process at the cost of having to iterate a few times in order to get an accurate solution.

Another revision of the classical Λ iteration aimed at speeding up its convergence, is based upon the use of certain quasi-invariant functions, the so-called iteration factors, whose iterative computation leads very quickly to the exact solution. In order to be good quasi-invariants, i.e. to change very little from one iteration to another, the factors have to be defined as the ratios of two homologous physical quantities. Such an idea appeared for the first time in the paper by Feautrier (1964). Its first realization was the variable (depth-dependent) Eddington factor (VEF) technique, developed in the papers by Auer and Mihalas (1970) and by Hummer and Rybicki (1971) for the solution of the monochromatic transfer problem in plane-parallel and spherical geometry, respectively. Defined as the ratio of the intensities angular moments, VEF is nearly independent on the trial source function getting quickly the correct value and thus providing a rapid convergence towards the exact solution.

A different approach to the solution of RT problems is introduced by the implicit integral method (IIM), developed by Simonneau and Crivellari (1993). Using an implicit treatment of the radiation field within a forward-elimination back-substitution scheme it solves global RT problem layer by layer with no need for storing and inverting matrices. It proved to be an efficient method in the solution of various RT problems, in the first place because its algorithm follows the physics of the RT process. Being at first developed for plane-parallel geometry, IIM is generalized to spherical geometry by Gros et al. (1997).

A new simple method which substantially accelerates the convergence of the ordinary Λ-iteration, while retaining its straightforwardness, is a forth-and-back implicit Λ iteration described in Atanacković-Vukmanović, Crivellari and Simonneau (1997), hereafter referred to as ACS. It uses an implicit representation of the source function in the computation of the mean intensities of the radiation field within a forth-and-back approach. A separate treatment of the propagation of the in-going and the out-going intensities of the radiation field (suggested by two separate sets of boundary conditions) enables to express intensity at any given point in any direction as a linear function of the unknown values of the source function and its derivative at that point and at the previous ones along the same direction. Iterative corrections of the coefficients of these *implicit* relations (implicit, as the source function is a priori unknown) instead of the unknown functions themselves, greatly accelerates the convergence of the direct iterative scheme.

In the ACS paper, the implicit Λ iteration (ILI) method is developed to solve non-LTE radiative transfer problems in stationary plane-parallel media. The aim of the present paper is to generalize the ILI method to spherical geometry. Although radiative transfer in plane parallel and spherically symmetric systems is the same from the physical point of view, the principal mathematical difficulty arises from the strong angular dependence of the radiation field produced by the curvature. At large radial distances, the intensity is concentrated within a narrow cone around the outward radial direction ("peaking effect"). The number of specific RT equations necessary to describe the strong anisotropy of the radiation field is exceedingly high, making the solution of the RT problems with spherical symmetry more complicated than that in plane–parallel geometry.

For the sake of an easier presentation of the ILI method when applied to spherical geometry we shall limit ourselves to the case of a monochromatic scattering in a spherical atmosphere. This simple case has been treated by several authors and serves as a good benchmark for testing the quality of the results.

2. RADIATIVE TRANSFER IN SPHERICAL MEDIA

We shall consider a stationary stellar atmosphere consisting of homogeneous spherical layers (whose physical properties vary only with radial distance r). For the numerical description of the radiation transport through such an atmosphere a discrete set of radii $\{r(L)\}$, $L = 1, N$ is required. Let the upper boundary surface of the atmosphere be at radius $r(1)$ and the lower one at radius $r(N)$ (see Fig. 1). The radius $r(N)$ is to be chosen so that the radiation field at that point be highly isotropic. It is customary to take $r(1)$ as the origin of the mean optical

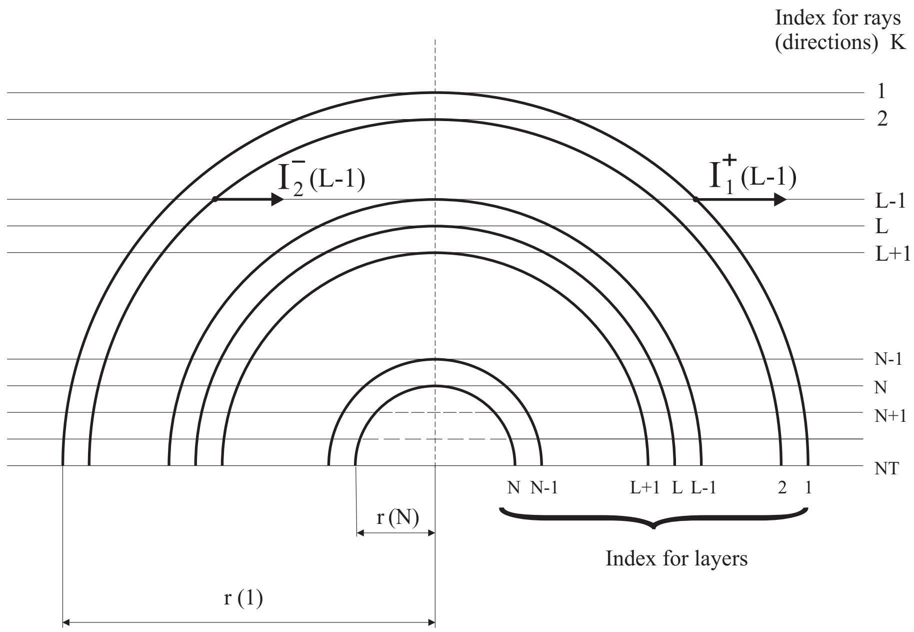

Fig. 1. *Discrete mesh of radii* $\{r(L)\}, L = 1, N$ *and a grid of rays (directions)* $K = 1, NT$ *that are used for the solution of the RT equation;* $I_L^{\pm}(K)$ *denote the in–going and out–going intensities along the direction* K *at any point* L.

depth scale along the radial direction τ, defined by

$$\tau(r) = \int_r^{r(1)} \chi(r')dr', \tag{1}$$

where the opacity $\chi(r)$ is assumed to be known. Hence, with the known opacity $\chi(\mathrm{r})$ we can compute the set of radial optical depths $\tau(L) = \tau[r(L)]$ so that $\tau(1) = 0$ and that $\tau(L)$ increases with depth.

In order to evaluate the radiation field in spherical media, one can use the radiative transfer (RT) equation either in partial differential or in ordinary differential form. RT equation written in polar coordinates reads:

$$\mu\frac{\partial I(r,\mu)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I(r,\mu)}{\partial \mu} = -\chi(r)[I(r,\mu) - S(r)], \tag{2}$$

where μ is the cosine of the angle between the outward radius vector and the direction of propagation of radiation at radius r, $I(r,\mu)$ is the specific intensity of the radiation field at point r and in the direction μ, $\chi(r)$ being the volume opacity coefficient and $S(r)$ the source function.

If we consider radiative transfer along a straight ray, RT equation has an ordinary differential equation form:

$$\pm\frac{dI^{\pm}}{d\tau} = I^{\pm} - S, \tag{3}$$

where τ is the optical depth along the ray due to the volume opacity coefficient χ (see eq. (6)), whereas $I^{\pm}$ is the specific intensity in the two directions along the ray.

Here we shall use the ordinary differential equation form and the discrete set of rays like the one used by Gros et al. (1997). The solution of RT equation (the computation of the intensities) is performed ray-by-ray along the set of directions tangent to the spherical layers corresponding to the discrete set $\{r(L)\}$ as well as along a few additional so-called core rays that intersect the inner boundary surface (see Fig. 1). The latter set of rays is added for better description of the inner boundary condition. These rays have impact parameters, i.e. closest approach to the center, $r(L)$; $L = N+1, NT$ where $r(NT) = 0$.

The intensities propagated along a ray K are computed in the points where the ray intersects the spherical shells of radii $r(L)$, with $L < K$. In these points the ray forms angles θ_K^L with the local outward radial directions whose cosines μ_K^L are given by

$$\mu_K^L = \sqrt{1 - \frac{r(K)^2}{r(L)^2}}. \tag{4}$$

According to customary terminology and notation, the intensity at point L propagating in the direction of increasing optical depth along a ray K (in-going intensity) is denoted as $I_L^-(K)$, whereas that propagating in the direction of decreasing optical depth (out-going intensity) as $I_L^+(K)$.

For the computation of the in-going intensities $I_L^-(K)$ we need the values of the incident intensities on the upper boundary surface $I_1^-(K), K = 1, NT$ (usually zero). For the outgoing intensities $I_L^+(K)$ propagating along the rays with impact parameters $r(L)$; $L = 1, N$, the initial conditions are $I_L^+(L) = I_L^-(L)$, as the radiation field is symmetric about the point $K = L$ at which $\mu_K^L = \mu_L^L = 0$. For the rays that intersect the core, the intensities $I_N^+(K), K = N+1, NT$ incident on the inner boundary surface must be given.

Thus, given the intensity $I_L(K)$ at point L propagating along a ray K the intensity $I_{L+1}(K)$ at point $L+1$ on the same ray K can be expressed using the integral form of the radiative transfer equation:

$$\begin{aligned} I_{L+1}(K) = \quad & I_L(K)e^{-\Delta\tau_{L,L+1}(K)} + \\ + \int_{\tau_L(K)}^{\tau_{L+1}(K)} \quad & S[\tau(K)]e^{-[\tau_{L+1}(K)-\tau(K)]}d\tau(K) \ . \end{aligned} \tag{5}$$

Here, $d\tau(K)$ is the differential optical distance along the ray K given by:

$$d\tau(K) = \chi[z(K)]dz(K), \tag{6}$$

where z is the geometrical distance along the ray. It can also be expressed in terms of differential radial optical depth $d\tau$ as follows:

$$d\tau(K) = \frac{d\tau}{\mu}. \tag{7}$$

From Eq. (5) we see that for the computation of the specific intensities, apart from the given boundary conditions, we need the values of the source function $\{S_L\}$ over the given mesh of depth points. However, the source function is not a priori known, as in general it contains the scattering integral. In the case of monochromatic scattering that we shall consider in this paper, this is the integral of the specific intensity of the radiation field over directions J_L given by:

$$J_L = \frac{1}{2}\int_{-1}^{1} I_L(\mu)d\mu. \tag{8}$$

In this case the source function is expressed as a linear function of the mean intensity of the radiation field J_L:

$$S_L = \varepsilon B_L + (1-\varepsilon)J_L, \tag{9}$$

where ε is the branching ratio between the thermal contribution B_L and the scattering integral J_L. Hence, all the specific intensities, i.e. all the specific RT Eq. (3), are coupled by the scattering term (8), that is, in practice, replaced by a finite sum of specific intensity values:

$$J_L = \frac{1}{2}\Sigma_{K=L}^{NT} W_L(K)[I_L^+(K) + I_L^-(K)]. \tag{10}$$

Here, $W_L(K)$ are the quadrature weights, that can be determined once we choose a functional representation for $I_L(\mu)$.

3. IMPLICIT Λ ITERATION

An efficient approach to the above mentioned problem described in the previous section is the implicit Λ iteration (ILI) method. Its good convergence properties are due to the following ideas. Similarly to integral methods, ILI uses the fact that although the values of the radiation field are unknown, its behavior can be easily represented by using the integral form of the RT equation. However, contrary to the 'global' implicit scheme of the integral methods, where the mean intensity of the radiation field at one point is expressed as a linear function of the unknown values of the source function at all the other points, the ILI scheme can be regarded as 'local'. Namely, a separate treatment of the in-going and the out-going intensities of the radiation field within a forth-and-back approach enables to derive *implicit* relations for the corresponding mean intensities at each spherical layer $L = 1, N$, in terms of the source function and its derivative in that layer. These relations are in general of the form:

$$J_L^{\pm} = a_L^{\pm} + b_L^{\pm}S_L + c_L^{\pm}S_L'. \tag{11}$$

According to the idea of the iteration factors, iterative computation of the coefficients of these *implicit* 'closure' relations between the unknown functions ($J^{\pm}$ and S) gives rise to a substantial acceleration of the ordinary iterative procedure.

Let us now derive relations (11) starting with the integral form of RT Eq. (5), i.e. considering the variation of the specific intensities between the layers L and $L+1$ along the direction K tangent to the sphere of radius $r(K)$, for which it holds that $K \geq L+1$. By assuming a piecewise polynomial behavior for the source function $S[\tau(K)]$ between points L and $L+1$ along the direction K, we can rewrite the integral in Eq. (5) in terms of the values of the source function and its first derivative at these two points, so that we have

$$\begin{aligned} I_{L+1}^-(K) \quad = \quad & I_L^-(K)e^{-\Delta\tau(K)} + p_L^-(K)S_L \\ & + p_{L+1}^-(K)S_{L+1} + q_L^-(K)[\frac{dS}{d\tau(K)}]_L \\ & + q_{L+1}^-(K)[\frac{dS}{d\tau(K)}]_{L+1} \end{aligned} \tag{12a}$$

for the in-going intensities (from L to $L+1$), and

$$\begin{aligned} I_L^+(K) \quad = \quad & I_{L+1}^+(K)e^{-\Delta\tau(K)} + p_L^+(K)S_L \\ & + p_{L+1}^+(K)S_{L+1} + q_L^+(K)[\frac{dS}{d\tau(K)}]_L \\ & + q_{L+1}^+(K)[\frac{dS}{d\tau(K)}]_{L+1}, \end{aligned} \tag{12b}$$

for the out-going intensities (from $L+1$ to L).

The coefficients $p^{\pm}$ and $q^{\pm}$ depend only on the known optical distance $\Delta\tau^{L,L+1}(K)$ between the two consecutive points L and $L+1$ along the ray K. The form of the coefficients depends on the assumed functional representation for $S(\tau)$. For the cubic piecewise approximation it holds that:

$$p^-_L = p^+_{L+1} = \frac{6}{\Delta^2} - \frac{12}{\Delta^3} - e^{-\Delta}(1 - \frac{6}{\Delta^2} - \frac{12}{\Delta^3})$$

$$p^-_{L+1} = p^+_L = 1 - \frac{6}{\Delta^2} + \frac{12}{\Delta^3} - e^{-\Delta}(\frac{6}{\Delta^2} + \frac{12}{\Delta^3})$$

$$q^-_L = -q^+_{L+1} = \frac{2}{\Delta} - \frac{6}{\Delta^2} + e^{-\Delta}(1 + \frac{4}{\Delta} + \frac{6}{\Delta^2})$$

$$q^-_{L+1} = -q^+_L = -1 + \frac{4}{\Delta} - \frac{6}{\Delta^2} + e^{-\Delta}(\frac{2}{\Delta} + \frac{6}{\Delta^2})$$

where $\Delta = \Delta\tau(K)$.

By using Eq. (7), we have

$$\Big[\frac{dS}{d\tau(K)}\Big]_L = \mu^L_K \Big[\frac{dS}{d\tau}\Big]_L = \mu^L_K S'_L \qquad (13)$$

and

$$\Big[\frac{dS}{d\tau(K)}\Big]_{L+1} = \mu^{L+1}_K \Big[\frac{dS}{d\tau}\Big]_{L+1} = \mu^{L+1}_K S'_{L+1} \qquad (14)$$

where primes denote the derivatives with respect to the radial optical depth τ. Inserting (13) and (14) into Eqs. (12) and eliminating S'_L according to:

$$S'_L = \frac{2}{\Delta\tau}(S_{L+1} - S_L) - S'_{L+1} \qquad (15)$$

the following expressions are obtained:

$$I^-_{L+1}(K) = a^-_{L+1}(K) + b^-_{L+1}(K)S_{L+1} + c^-_{L+1}(K)S'_{L+1} \qquad (16a)$$

and

$$I^+_L(K) = a^+_L(K) + b^+_L(K)S_L + c^+_L(K)S'_L \qquad (16b)$$

that relate the in-going, i.e. the out-going specific intensities with the source function and its derivative at the corresponding layer. The coefficients of Eq. (16a) are:

$$\begin{aligned} a^-_{L+1}(K) &= I^-_L(K)e^{-\Delta\tau(K)} + \\ &+ \Big[p^-_L(K) - \frac{2\mu^L_K q^-_L(K)}{\Delta\tau}\Big]S_L \\ b^-_{L+1}(K) &= p^-_{L+1}(K) + \frac{2\mu^L_K q^-_L(K)}{\Delta\tau} \\ c^-_{L+1}(K) &= \mu^{L+1}_K q^-_{L+1}(K) - \mu^L_K q^-_L(K) \qquad (17a) \end{aligned}$$

whereas those of Eq. (16b) are:

$$\begin{aligned} a^+_L(K) &= I^+_{L+1}(K)e^{-\Delta\tau(K)} + \\ &+ \Big[p^+_{L+1}(K) + \frac{2\mu^L_K q^+_L(K)}{\Delta\tau}\Big]S_{L+1} \\ b^+_L(K) &= p^+_L(K) - \frac{2\mu^L_K q^+_L(K)}{\Delta\tau} \\ c^+_L(K) &= \mu^{L+1}_K q^+_{L+1}(K) - \mu^L_K q^+_L(K). \qquad (17b) \end{aligned}$$

The coefficients given by (17a) should be computed and stored in the forward step of each iteration to be used later during the backward step for updating the current solution.

Finally, in order to derive the relations (11) for the in-going and the out-going mean intensities, we have to perform numerical integration of the pertaining specific intensities over all directions K. Assuming that $I_L(\mu)$ varies linearly with μ between the subsequent values $I^{\pm}_L(K)$ at μ^L_K ($0 \le \mu^L_K \le 1$), where $\mu^L_L = 0$ and $\mu^L_{NT} = 1$, the weight coefficients are given by:

$$W_L(K) = \begin{cases} \frac{1}{2}(\mu^L_2 - \mu^L_1), & K = 1 \\ \frac{1}{2}(\mu^L_{K+1} - \mu^L_{K-1}), & K = 2, NT-1 \\ \frac{1}{2}(\mu^L_{NT} - \mu^L_{NT-1}), & K = NT \end{cases} \qquad (18)$$

Let us now consider the forward and the backward steps in more detail.

3.1 The forward process

We start each iteration with the layer $L = 1$ and the given upper boundary condition:

$$I^-_1(K) = 0; \quad K = 1, NT. \qquad (19)$$

According to (19), the coefficients (17a) for the first layer are get equal to zero. We proceed with the computation and storage of the coefficients (17a) at all subsequent depth points. Let us note here that the coefficients $a^-_L(K), L = 2, N$ are to be computed with the old value (known from the previous iteration) of the source function S^o. In order to match the value of the old source function S^o used for the computation of $a^-_L(K)$ with the updated (new) value S^n at point L, we scale the function S^o by the factor S^o/S^n and re-write Eq. (16a) for layer L in the form:

$$I^-_L(K) = [\frac{a^-_L(K)}{S^o_L} + b^-_L(K)]S_L + c^-_L(K)S'_L. \qquad (20)$$

Here,

$$\begin{aligned} a^-_L(K) &= I^-_{L-1}(K)e^{-\Delta\tau^{L-1,L}(K)} + \\ &+ \Big[p^-_{L-1}(K) - \frac{2\mu^{L-1}_K q^-_{L-1}(K)}{\Delta\tau}\Big]S^o_{L-1} \end{aligned}$$

is computed with the old source function S^o, $b^-_L(K)$ and $c^-_L(K)$ depend only on the known optical distances $\Delta\tau^{L-1,L}(K)$, whereas S_L and S'_L are the yet unknown source function and its $\tau-$ derivative at layer L.

After numerical integration of (20) over all directions we obtain the linear relation

$$J_L^- = b_L^- S_L + c_L^- S_L', \tag{21}$$

representing *implicitly* the values of the in-going mean intensities. Hence, in the forward process, ILI differs from the classical Λ iteration that recalculates J_L^- from the old (known) source function $S^o(\tau)$, in the use of the old source function only to compute and store the coefficient $a_L^-(K)$ of the linear relation (20) at each radial optical depth point τ_L (L=1,N) for further use in the backward process of computation of the new values $S^n(\tau)$.

3.2 The backward process

Now, we proceed from the bottom layer $N-1, N$ for which the incident up-going intensities $I_N^+(K)$ are given (explicitly or implicitly, e.g. by using the diffusion approximation), J_N^+ is known too. Inserting in the expression for the total mean intensity (10), re-written for $L = N$ as

$$J_N = \frac{1}{2}[J_N^+ + J_N^-], \tag{22}$$

together with the Eq. (21) for J_N^-, where S_N' can be eliminated by assuming linear behavior of $S(\tau)$ in the last layer

$$S_N' = S_{N-1}' = (S_N - S_{N-1})/\Delta\tau, \tag{23}$$

J_N results in a linear combination of S_N and S_{N-1}. A similar linear relation for J_{N-1} can be obtained by using Eq. (16b) after integration over directions and Eq. (21) for the layer $N-1$. Then new values S_{N-1}^n and S_N^n can be easily derived. The derivatives S_N' and S_{N-1}' are to be computed from Eq. (23) and the values $I_{N-1}^+(K)$ obtained from Eq. (16b). The new values of S_{N-1}, S_{N-1}' and $I_{N-1}^+(K)$ can then be used as known boundary conditions for the next upper layer $(N-1, N-2)$. Namely, these values are used to compute the coefficients of Eq. (16b) at the next point L $(= N-2)$ which, after the use of Eq. (15), can be re-written for any layer L in the form:

$$I_L^+(K) = a_L^+(K) + b_L^+(K) S_L. \tag{24}$$

Here, the coefficient $b_L^+(K)$ depends only on the known optical distance, whereas the coefficient $a_L^+(K)$ is computed with the updated values of $I_{L+1}^+(K)$, S_{L+1} and S_{L+1}'. By integrating (24) over directions, we obtain the linear relation

$$J_L^+ = a_L^+ + b_L^+ S_L \tag{25}$$

at each radial depth point L. Using Eq. (15) to eliminate S_L' from Eq. (21), i.e. to express S_L' in terms of the yet unknown value of S_L, and the known values of S_{L+1} and S_{L+1}', we can write J_L^- as a linear function of S_L only. Therefore, we obtain the linear relation between J_L and S_L:

$$J_L = a_L + b_L S_L. \tag{26}$$

The computation of the coefficients a_L and b_L of Eq. (26) and its solution together with Eq. (9) to get a new source function S_L, is performed during the backward process layer by layer to the surface.

4. RESULTS AND DISCUSSION

Here we test the above described procedure on the solution of monochromatic scattering problem in a spherical atmosphere, as given in Avrett and Loeser (1984). The source function is of the form:

$$S = \alpha J + (1-\alpha)B$$

with a scattering coefficient $\alpha = 0.5$ and $B = 1$. The inner and outer boundaries are at $r(N) = 1$ and $r(1) = 30$, respectively, measured in units of the stellar radius. The opacity law $\chi(r)$ is given by C/r^2, where the constant C is evaluated from the requirement that the total radial optical thickness of the atmosphere is 4. Hence, the radial optical depth is given by:

$$\tau(r) = \frac{120}{29}\left(\frac{1}{r} - \frac{1}{30}\right).$$

This problem was solved by the implicit integral method in the paper by Gros et al. (1997), and the solutions were compared with those of Mihalas et al. (1975), Rogers (1981) and Avrett and Loeser (1984). In Table 1 we listed them, together with our results obtained by ILI method (the last column).

It can be seen that the relative differences of the solutions are about 1%. As pointed out in GCS, the differences between the results may arise due to different discretization in r (or in τ) and different approximations used to describe the behavior of $S(\tau)$ between any two consecutive depth points. In order to investigate both the accuracy and the convergence properties of the method, a number of tests were run with various discretization in τ (i.e. in r). As in the plane–parallel case the best discretization is the one that uses logarithmic spacing in the radial optical depth steps (with 5-10 points per decade).

Fig. 2 shows the evolution with iterations of the mean intensity of the radiation field, computed by the forth-and-back implicit Λ iteration method. Starting from $S = B = 1$ we get the exact solution, practically, already in the second iteration. An additional iteration was needed to meet the tolerance criterion (to stop the run of iterations) that the greatest relative correction (at all radial optical depths) between two successive iterations be less than 1%. The relative correction of 0.1% has been achieved in the fifth iteration.

Table 1. Values of $J(r)$ obtained by different authors (Mihalas et al. (MKH), Rogers (R), Avrett and Loeser (AL), Gros et al. (GCS))

r	$J(r)^{MKH}$	$J(r)^{R}$	$J(r)^{AL}$	$J(r)^{GCS}$	$J(r)^{this\,work}$
30	0.0636	0.0638	0.0642	0.0637	0.0638
29.6	0.0671	0.0674	0.0676	0.0675	0.0674
29	0.0716	0.0716	0.0721	0.0718	0.0718
28	0.0784	0.0785	0.0791	0.0787	0.0787
26	0.0920	0.0921	0.0928	0.0922	0.0924
24	0.106	0.107	0.108	0.1065	0.1068
20	0.141	0.140	0.142	0.1404	0.1411
16	0.182	0.187	0.190	0.1866	0.1877
12	0.260	0.255	0.261	0.2554	0.2573
9	0.345	0.336	0.340	0.3361	0.3387
6	0.493	0.468	0.476	0.4690	0.4724
4	0.638	0.615	0.627	0.6166	0.6206
2	0.864	0.849	0.859	0.8510	0.8553
1	0.964	0.961	0.986	0.9844	0.9779

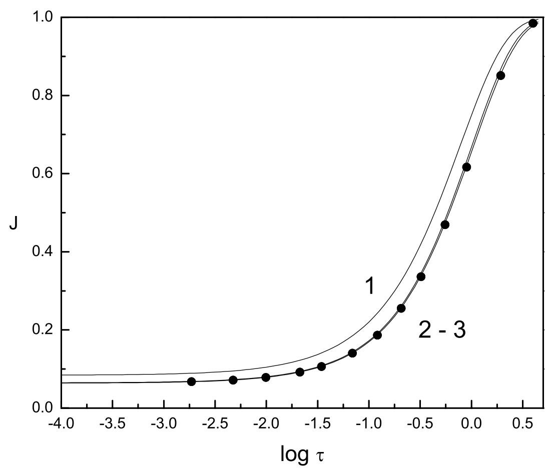

Fig. 2. *Mean intensity vs. optical depth. The solid lines, labelled with the corresponding iterations number, correspond to forth-and-back implicit Λ iteration. The circles mark the solution obtained by Gros et al. (1997).*

The obtained results confirm a good quality and fast convergence properties of the implicit Λ iteration method when applied to spherically symmetric media.

Acknowledgements – The author is grateful to the referee, prof. M. Vukićević–Karabin, for useful suggestions that helped improve the manuscript. This work was realized within the project No. 1191 ("Stellar physics") supported by the Ministry of Science, Technology and Development of the Republic of Serbia.

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ДВОСМЕРНО ИМПЛИЦИТНА Λ ИТЕРАЦИЈА ЗА РЕШАВАЊЕ ПРЕНОСА ЗРАЧЕЊА У СФЕРНО-СИМЕТРИЧНИМ СРЕДИНАМА

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Оригинални научни рад

Двосмерно имплицитна Λ итерација је развијена за решавање проблема преноса зрачења у план-паралелним срединама, када су једначине преноса зрачења међусобно повезане интегралом расејања који се јавља у изразу за функцију извора (Atanacković-Vukmanović, Crivellari i Simonneau 1997). Захваљујући имплицитном представљању функције извора у рачуну средњих интензитета при двосмерном третирању интензитета који се простиру у супротним правцима, имплицитна Λ итерација (ИЛИ) се показала као врло ефикасан метод у решавању линеарних и нелинеарних проблема преноса зрачења. У овом раду ИЛИ метода је уопштена и примењена на проблеме преноса зрачења у сферно-симетричним срединама. Приказани су резултати за монохроматски пренос зрачења у сферно-симетричној атмосфери и упоређени са резултатима добијеним коришћењем других метода.