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THE PLASMA-SOLID TRANSITION: SOME IMPLICATIONS IN ASTROPHYSICS

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SUMMARY: Using a criterion proposed by Salpeter and standard solid-state physics, we have determined the Debye temperature of a solid in equilibrium with the electron gas surrounding it. The results obtained can have astrophysical applications in the determination of parameters of interstellar and interplanetary clouds.

1. INTRODUCTION

Phase transitions of various kinds have been, and still are, actively contributing to important changes occurring in the world around us. The aim of the present paper is a contribution to the study of the conditions for the occurrence of the so-called "plasma-solid (PS)" transition.

Simple physical reasoning shows that the PS transition must certainly occur in various astrophysical settings, such as in proto-planetary and proto-stellar clouds. On the one hand, it is widely known that the universe contains different kinds of plasmas. On the other hand, it is equally widely known that solid objects also exist in the universe. This implies that there must exist a physical region where a transition between the two regimes takes place. Note also that investigations of the PS transition could be interesting in non-astrophysical plasma physics, such as in considerations of a laser produced plasma in front of a metal target (*cf.* Dimitrijević and Konjević, 1980).

In a preliminary study of the PS transition (Čelebonović and Däppen, 2000), we have determined the conditions for its occurrence using two simple idealized systems: a pure Fermi-Dirac and a pure Bose-Einstein gas. The object of the present calculation is again a determination of the conditions for the occurrence of the PS transition, but this time in a more realistic model. Our starting point is the criterion for the occurrence of the PS transition (which was not given that name) proposed by Salpeter, 1961.

Salpeter's paper was devoted to a thorough discussion of a zero-temperature plasma. In the third part of that paper, he considered a system of positive ions of given charge and mass, rigidly fixed in the nodes of a perfect crystal lattice, and went on to *estimate* the zero point energy of the ions. He showed that the behavior of such a system can be described by the ratio

$$f_S = \frac{E_{z,p}}{E_C} \ , \tag{1}$$

where $E_{z,p}$ denotes the zero point energy of the ions and E_C is the Coulomb energy. According to the analysis of Salpeter (1961) a PS transition occurs for $f = 1$. Calculations reported in that paper were estimates, and strictly valid only for $T = 0$ K. In the calculations reported in the following, we have somewhat reformulated this criterion; namely, we have compared the energies per particle of a solid and the electron gas.

This means that in our formulation,the behaviour of the system is governed by the ratio

$$f_m = \frac{E_{p,s}}{E_{p,e}} \qquad (2)$$

where $E_{p,s}$ denotes the energy per particle of a solid, and $E_{p,e}$ is the energy per particle of the electron gas. The imposed the condition $f_m = 1$ defines a line (or a region) in the phase space of the system where the energies per particle in the two phases are equal. In standard phase transition theory one of the conditions needed for phase equilibrium in a multiphase system is the equality of the chemical potentials of its phases (for example, Landau and Lifchitz 1976). Salpeter's criterion is in fact equivalent to that statement, but under one supplementary condition - that the energy is independent of the particle number. Moving off this line (or out of the region) defined by $f_m = 1$ implies the occurence of a phase transition. Our calculations go beyond the original assumptions of Salpeter in at least two aspects: (i) we have not used estimates but exact calculations, and (ii) our results take into account the influence of temperature.

2. CALCULATIONS

Finding a general expression for the energy of a real solid that takes into account most (if not all) of its characteristics is a formidable problem in solid state physics (for example Born and Huang 1968, or Davydov 1980). The complexity of the problem is caused by several factors:the frequency distribution of the oscillations of particles around their equilibrium positions,their mutual interactions and their large number (which is of the order of 10^{23}).

Various approximations to the complete solution exist; one of them is the Debye model of a solid. Within the Debye model, the energy per mole of a solid is given by the following expression (Born and Huang 1968)

$$E = N \left\{ u(v) + 9 n k_B T \left[\left(\frac{T}{\theta} \right)^3 \int_0^{\theta/T} \left(\frac{1}{2} + \frac{1}{\exp(\xi) - 1} \right) \xi^3 d\xi \right] \right\}, \qquad (3)$$

where N denotes the number of elementary cells in a mole of the material, n the number of particles per elementary cell, $u(v) = \frac{U}{N}$ = the static lattice energy per cell, θ the Debye temperature, k_B the Boltzmann constant. In the following we will use the convention $k_B = 1$. The second integral in Eq. (3) can be solved as (Abramowitz and Stegun 1972)

$$I_1 = \int_0^x \frac{t^n dt}{e^t - 1} = x^n \left[\frac{1}{n} - \frac{x}{2(n+1)} + \sum_{k=1}^{\infty} \frac{B_{2k} x^{2k}}{(2k+n)(2k)!} \right] \qquad (4)$$

under the condition $n \succeq 1, |x| \prec 2\pi$.These conditions are in Abramowitz and Stegun (1972) introduced for purely mathematical reasons. Physically, $|x| \prec 2\pi$ implies $T > \frac{\theta}{2\pi}$.This will be used later in this paper to express the Debye temperature of a solid in equilibrium with a plasma, as a function of the number density. Here, the symbol B_{2k} denotes Bernoulli's numbers. Taking the static lattice energy per cell as the zero point of the energy scale, it follows from Eq. (3) that the energy per particle of a solid within the Debye model is given by

$$E_{p,s} = 9T \left(\frac{T}{\theta} \right)^3 I \qquad (5)$$

where

$$I = \frac{1}{2} \int_0^{\theta/T} \xi^3 d\xi + \int_0^{\theta/T} \frac{\xi^3 d\xi}{e^{\xi} - 1} \qquad (6)$$

Using Eq. (4) in Eq.(6), one finally gets

$$\left(\frac{T}{\theta} \right)^3 I = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k+3)(2k)!} \left(\frac{\theta}{T} \right)^{2k} \qquad (7)$$

which implies

$$E_{p,s} = 3T \left[1 + 3 \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k+3)(2k)!} \left(\frac{\theta}{T} \right)^{2k} \right] \qquad (8)$$

Expanding explicitly it follows that

$$E_{p,s} \cong 3T \left[1 + \frac{1}{20} \left(\frac{\theta}{T} \right)^2 - \frac{1}{1680} \left(\frac{\theta}{T} \right)^4 + .. \right] . \qquad (9)$$

The energy per particle of a Fermi-Dirac gas of particles of number density n mass m and spin s is given by (Landau and Lifchitz 1976)

$$E_{p,e} = \frac{g m^{3/2}}{2^{1/2} n \pi^2 \hbar^3} F_{3/2}(\beta\mu) \qquad (10)$$

All the symbols in this equation have their usual meaning. In particular, $g = 2s + 1$, and $F_{3/2}(\beta\mu)$ is the $k = 3/2$ case of a Fermi integral

$$F_k(\beta\mu) = \int_0^{\infty} \frac{\epsilon^k d\epsilon}{1 + \exp[\beta(\epsilon - \mu)]} . \qquad (11)$$

Analytical expansions of this type of integrals into power series are known in the literature (such as Čelebonović 1998). The ratio f_m as defined in Eq. (2)

can be formed from Eqs. (8) and (10). After some algebra one gets the following general expression for this ratio:

$$f_m = (32^{1/2} n\pi^2\hbar^3 T)/(gm^{3/2} F_{3/2}(\beta\mu))$$
$$[1 + 3\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k+3)(2k)!}\left(\frac{\theta}{T}\right)^{2k}] \tag{12}$$

This result is of interest by being mathematically general (within a particular model of a solid). However, due to this generality hardly any physically useful conclusions can be drawn from it. Limiting the sum in Eq. (12) to terms containing $(\frac{\theta}{T})^2$ and inserting a known power series expansion for the Fermi integral leads to the following approximation for the ratio f_m:

$$f_m = \frac{9n}{2KT}\frac{20T^2 + \theta^2}{(12A^2 n^{4/3} + 5\pi^2 T^2)} \tag{13}$$

The constant A in the preceeding equation occurs because of the expression for the chemical potential of the electron gas at $T = 0$ K : $\mu_0 = An^{2/3}$, and it is given by $A = \left(3\pi^2\right)^{2/3}\hbar^2/2m$.The symbol K denotes the following combination of constants: $K = \frac{gm^{3/2}}{2^{1/2}\pi^2\hbar^3}$.

Imposing the condition $f_m = 1$ on Eq.(13), it becomes possible to determine the conditions for the occurence of a PS transition. Solving Eq. (13) for the Debye temperature θ gives

$$\theta = \frac{1}{3}\sqrt{[2KT/n]}\sqrt{[5\pi^2 T^2 + 12A^2 n^{4/3}]}$$
$$\sqrt{[1 - \frac{90nT}{(5\pi^2 T^2 + 12A^2 n^{4/3})}]} \tag{14}$$

We have thus obtained the equation of state of a system undergoing a PS transition. This equation links the relevant parameters of a plasma (number density and temperature) with those of the solid (Debye temperature) which is in phase equilibrium with it.

3. DISCUSSION

Apart from being interesting from the point of view of pure statistical physics, this result can find astrophysical applications in studies of dust and gas clouds. Recent examples of observational studies of such clouds are for example those of Jessop and Ward-Thomson (2000) and Grün, Krüger and Landgraf (2000).The particle number density and the temperature of an astrophysical plasma can be determined from observation using laboratory methods, such as the analysis of spectral line broadening (e.g. Dimitrijević 1996) , or in some cases, *in-situ* measurements from space probes. Inserting these values into Eq. (14) affords the possibility to determine the Debye temperature of a solid which is in equilibrium with the plasma surrounding it. A real physical example of such a system could be a condensing protoplanetary cloud, such as the one around the star β Pictoris.

Consider Eq. (14) in the limiting case $T \to 0$. Astrophysically, this limit corresponds to an isolated low temperature interstellar cloud, or to a protoplanetary disk sufficiently distant from the central star. Mathematically, this means that all functions of T in Eq. (14) can be expanded into series with T as a small parameter. Keeping only the first terms in these expansions, one finally gets the following expression for the Debye temperature:

$$\theta = \frac{2}{3}\sqrt{[2KT/n]}3^{1/2}An^{2/3}[1 - \frac{45}{12}\frac{T}{KA^2 n^{1/3}}$$
$$(1 - \frac{5}{12}\frac{\pi^2 T^2}{A^2 n^{4/3}})] \tag{15}$$

All the constants appearing in Eq. (15) have been previously defined.

Inserting the values of all the constants which appear in Eq. (15), and retaining only the biggest term, one finally arrives at

$$\theta = 67387.5\, T^{5/2} n^{-3/2} \tag{16}$$

This expression is the final approximation derived in this paper for the Debye's temperature of a solid in equilibrium with the plasma surrounding it. Its applicability is limited by the condition $\frac{\theta}{T} \prec 2\pi$, which is built-in in Eq. (4). Introducing this condition in Eq. (16) leads to:

$$T \prec 0.00205621 \times n \tag{17}$$

This is the physical limit of applicability of Eq. (16). It is physically expectable, because of the limitations inherent to Debye's model on which the calculation leading to Eq. (16) is based. Expression (16) has two possible applications. It can be used for the calculation of θ if the temperature and the number density of the cloud are known. Recent observation shows that the temperature in the cold protostellar clouds in our galaxy is $10 < T[\mathrm{K}] < 20$ (Cesarsky *et al.* 2000). Typical values of the density are in the interval $10^4 < n(\mathrm{cm}^{-3}) < 10^5$. Plotting the behaviour of Eq. (16) within these limits shows that $\theta \leq 1600$ K. In a similar way, the pair of values $n = 1.25 \times 10^4 \mathrm{cm}^{-3}$, $T = 15$ K would lead to $\theta = 630$ K. Such a value is only slightly higher than the experimental value for the element Si. Note that this last result could be useful for the interpretation of observations, because emission from silicate grains has indeed been detected (Cesarsky *et al.* 2000). Turning the argument around, if solid particles are observed in a cloud, and if their chemical composition can be determined, it becomes possible to calculate the Debye temperature from the principles of solid state physics. If, in addition, temperature is known from spectroscopy, one can determine the value of the chemical potential and finally the number density of a cloud from Eq. (16). Further work along these lines is in progress and will be discussed elsewhere.

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ФАЗНИ ПРЕЛАЗ ПЛАЗМА-ЧВРСТО ТЕЛО И ИМПЛИКАЦИЈЕ У АСТРОФИЗИЦИ

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Помоћу критеријума који је предложио Салпетер, и неких резултата из физике чврстог стања израчунали смо Дебајеву температуру чврстог тела у равнотежи са електронским гасом који га окружује. Резултати овог рада могу да нађу примену у одређивању параметара међузвезданих и међупланетарних облака.