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Preliminary report

ON THE GEODESICS FOR A SPHERICALLY SYMMETRIC DILATON BLACK HOLE

C. Blaga[1] and P. A. Blaga[2]

[1] *Astronomical Observatory, Cireşilor 19, 3400 Cluj-Napoca, Romania*

[2] *"Babeş-Bolyai" University, Faculty of Mathematics and Computer Sciences, 1, Kogălniceanu Street, 3400 Cluj-Napoca, Romania*

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SUMMARY: In this paper we shall investigate the timelike geodesics for an extremal, spherically symmetric, massless dilaton black hole, using an exact solution obtained by Gary Horowitz.

1. INTRODUCTION

In classical general relativity, the geometry of a static, charged, spherically symmetric black hole is described by the well-known Reissner-Nordström solution. However, if string theory is to be used for the description of nature, then, in the low energy limit of this theory, the action includes, besides the pure gravitational part, a minimally coupled scalar field, the dilaton. Horowitz (1993) showed that, by applying to the Schwarzschild solution a Harrison-like transformation, we can obtain a metric which is a solution of the Einstein-Maxwell-dilaton field equations and a not very difficult investigation reveals the fact that it is a static, spherically symmetric solution, corresponding to a charged, massless dilaton. The line element for this solution is given by

$$\begin{aligned} ds^2 = &-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+ \\ &+r\left(r-\frac{Q^2}{M}\right)\left[d\theta^2+\sin^2\theta d\varphi^2\right] \end{aligned} \tag{1}$$

In a previous paper (Blaga, Blaga, 1996), we were able to show that the geodesics equation for this metric is separable and we obtained the following generating function

$$S=-\frac{1}{2}\delta_1-Et+L_z+\int^r\sqrt{\frac{R}{\Delta}}dr+\int^\theta\sqrt{\Theta}d\theta, \tag{2}$$

in curvature coordinates. Here

$$\delta_1=\begin{cases}-1 & \text{for timelike geodesics}\\ 0 & \text{for null geodesics}\\ 1 & \text{for spacelike geodesics}\end{cases}$$

E is a constant, identified to the total energy of the particle moving on the geodesics.
L_z is the kinetic energy of the particle.

$$R=-\delta_1 r\left(r-\frac{Q^2}{M}\right)+\frac{r\left(r-\frac{Q^2}{M}\right)}{1-\frac{2M}{r}}E^2-\mathcal{Q} \tag{3a}$$

$$\Delta=r\left(r-\frac{Q^2}{M}\right)\left(1-\frac{2M}{r}\right) \tag{3b}$$

$$\Theta=\mathcal{Q}-\frac{L_z^2}{\sin^2\theta} \tag{3c}$$

$\mathcal{Q}$ is the fourth constant of the motion (besides δ_1, E and L_z), it is a separation constant and it was first introduced by Brandon Carter, when he studied the motion around a Kerr-Newman black hole.

2. THE GEODESIC EQUATIONS

Starting from the complete integral (2), by using the Hamilton-Jacobi theorem, we recast the geodesics equations in the following first order form:

$$d\lambda = \frac{r\left(r - \frac{Q^2}{M}\right)dr}{\sqrt{R}\sqrt{\Delta}} \tag{4a}$$

$$dt = \frac{r\left(r - \frac{Q^2}{M}\right)E}{\left(1 - \frac{2M}{r}\right)\sqrt{R}\sqrt{\Delta}}dr \tag{4b}$$

$$d\varphi = \frac{L_z}{\sin^2\theta\sqrt{\Theta}}d\theta \tag{4c}$$

$$\frac{dr}{\sqrt{\Delta}\sqrt{R}} = \frac{d\theta}{\sqrt{\Theta}}, \tag{4d}$$

allowing us the study of the motion in this field.

The quantity Q is related to the electrical charge of the black hole. Horowitz emphasized that we are actually dealing with a black hole (and not a naked singularity) only for $Q^2 \leq 2M^2$. We shall consider hereafter only the case $Q^2 = 2M^2$, referred to as *the extremal case*.

The timelike geodesics from the equatorial plane $\theta = \pi/2$ are described by the equations (4c) and (4d). †

Passing from the variable r to the variable $u = \frac{1}{r}$ we get:

$$\left(\frac{du}{d\varphi}\right)^2 = (2Mu-1)^2\left(-u + \frac{2M}{L_z^2}u + \frac{E^2-1}{L_z^2}\right) \equiv f(u), \tag{5}$$

or, passing back to r and performing the square root,

$$d\varphi = \pm\frac{L_z dr}{|r-2M|\sqrt{r^2(E^2-1)+2Mr-L_z^2}}. \tag{6}$$

It is now clear how to use the equation(6) to determine the behaviour of geodesics. For each set of values of the parameters (M, E, L_z) the only motions that are allowed are those for which the argument of the square root is (strictly) positive.

What is interesting to note is that in the case of the dilaton black hole, no geodesic actually is passing through the event horizon $r = 2M$, which means that no uncharged test particle can reach the singularity in a free fall. This is not a contradiction with the fact that we are dealing with a black hole, because the singularity can be reached in a non-geodesic motion.

We were able to integrate the equation (6) and we obtained the following results, for different values of the quantities:

$$a = 4M^2E^2 - L^2 \text{ and } \Delta = M^2 + L^2(E^2-1).$$

Using also the notations

$$I = \pm\frac{(\varphi - \varphi_0)}{L}, \quad x = \frac{1}{r-2M},$$
$$b = 2M(2E^2-1), \quad \text{and} \quad c = E^2 - 1,$$

we have the following cases:

(i) $a < 0$ and $\Delta > 0$ the solution has the form

$$r = 2M - \frac{\frac{2a}{b}}{1 + \frac{\sqrt{\Delta}}{b}\sin(\sqrt{-a}I)}, \tag{6.1}$$

(ii) $a = 0$, if $bx + c > 0$ the solution reads:

$$r = 2M + \frac{4b}{b^2I^2 - 4c}, \tag{6.2}$$

(iii) $a > 0$ and Δ arbitrary. The solution is

$$r = 2M + \frac{4\varepsilon a e^{\sqrt{a}I}}{(e^{\sqrt{a}I} - \varepsilon b)^2 - 4ac} \tag{6.3}$$

where

$$\varepsilon = sgn(2ax + b + 2\sqrt{a(ax^2+bx+c)})$$

It goes without saying, but has to be said that only selected parts of the curves described by the equations (6.1)–(6.3) are actually geodesics.

The general characteristics of the geodesics will be dealt with in a forthcomming paper. What we want here is simply to illustrate the shape of them for some set of parameters.

3. EXAMPLES

We selected for this paper a number of four sets of values for the parameters (M, E, L_z), to illustrate the shape of the timelike geodesics (see Figures 1–4).

† It is not difficult to see that a geodesics for which at a given moment t_0, $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$, does not leave the equatorial plane.

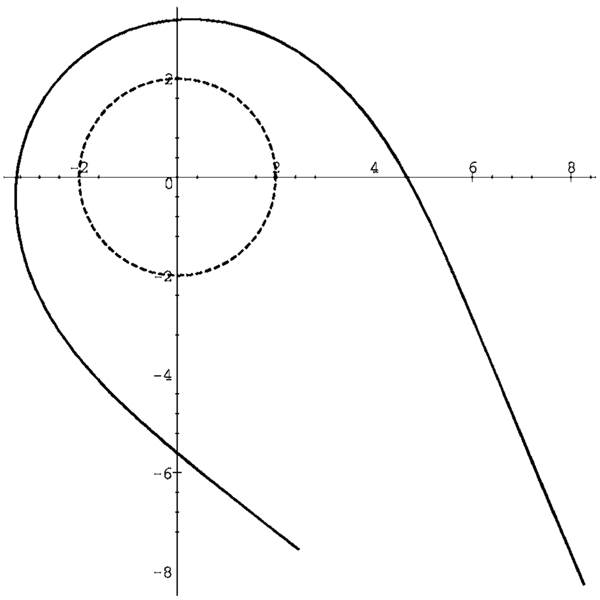

Fig. 1. $M = 1, E = \sqrt{2}, L_z = 4$ *(case (i))*.

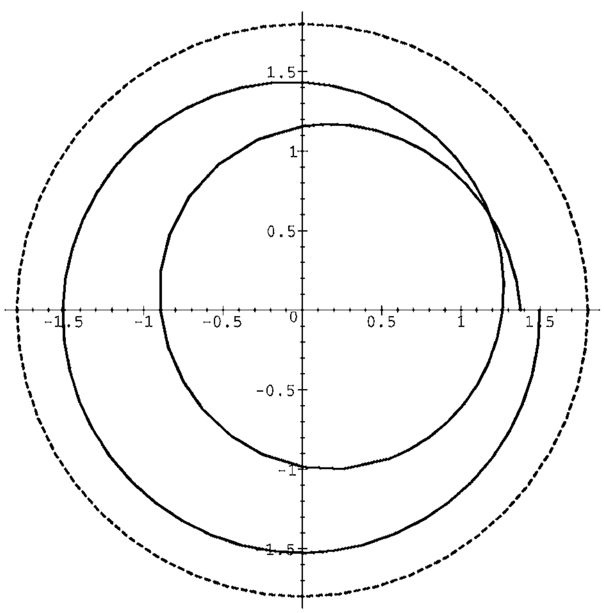

Fig. 2. $M = 0.9, E = 0.5, L_z = 1.5$ *(case (i))*.

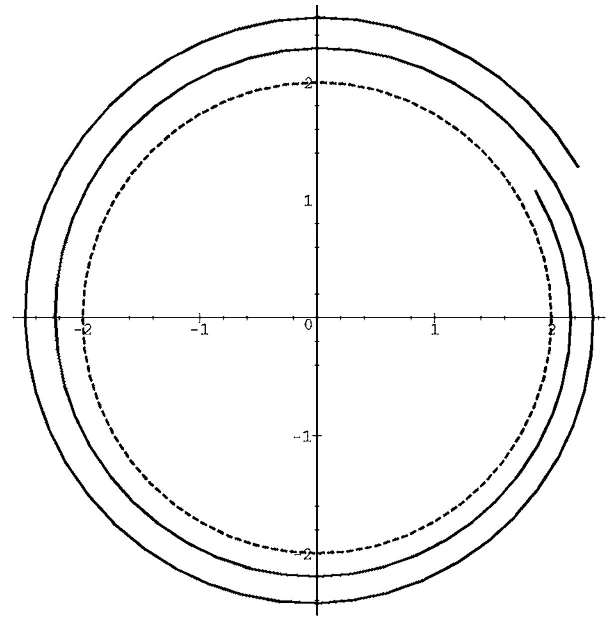

Fig. 3. $M = 1, E = 0.75, L_z = 1.5$ *(case (ii))*.

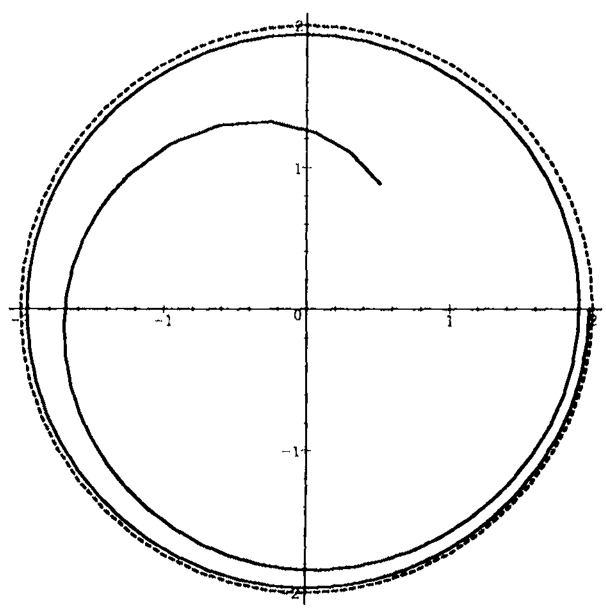

Fig. 4. $M = 1, E = 0.5, L_z = 1$ *(case (ii))*.

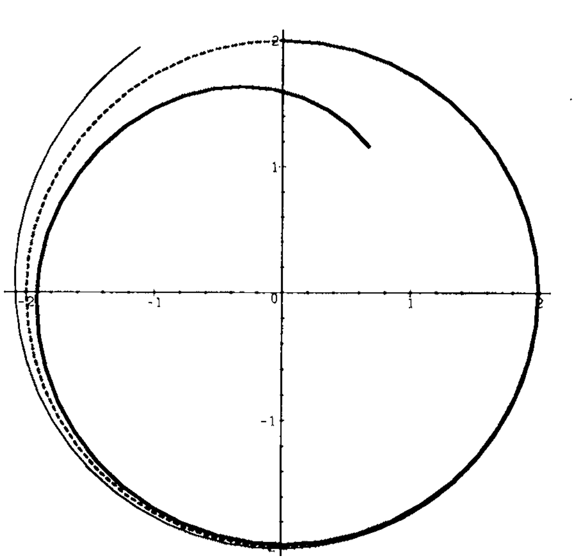

Fig. 5. $M = 1, E = \frac{1}{\sqrt{2}}, L_z = 1$ *(case (iii)).*

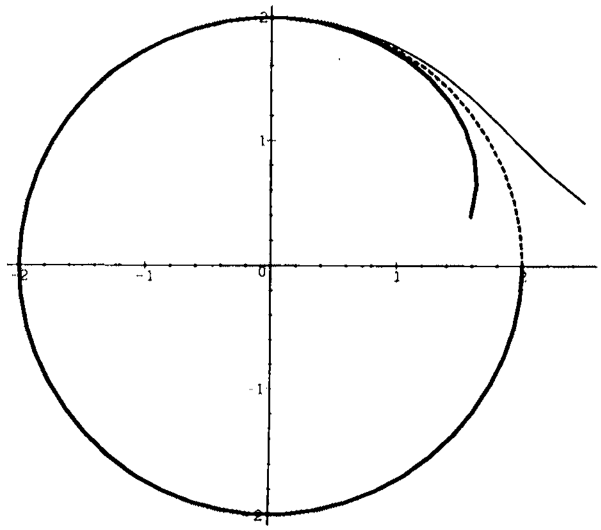

Fig. 6. $M = 1, E = 2, L_z = 2$ *(case (iii)).*

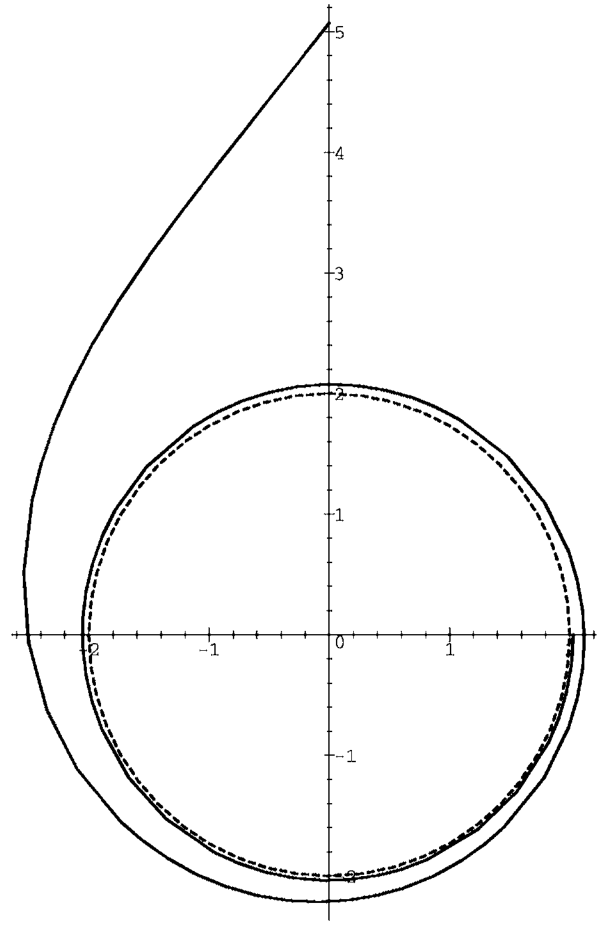

Fig. 7. $M = 1, E = 2, L_z = 4$ *(case (ii)).*

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О ГЕОДЕЗИЦИМА ЗА СФЕРНО СИМЕТРИЧНУ БЕЗМАСЕНУ ЦРНУ ЈАМУ

C. Blaga[1] and P. A. Blaga[2]

[1]*Astronomical Observatory, Cireşilor 19, 3400 Cluj-Napoca, Romania*

[2]*"Babeş-Bolyai" University, Faculty of Mathematics and Computer Sciences, 1, Kogălniceanu Street, 3400 Cluj-Napoca, Romania*

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Претходно саопштење

У овом раду испитујемо временске геодезијске линије у екстремној, сферно симетричној, безмасеној црној јами, користећи једно тачно решење које је добио Gary Horowitz.