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THE ZONAL SATELLITE PROBLEM. II. NEAR-ESCAPE FLOW

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SUMMARY: The study of the zonal satellite problem is continued by tackling the situation $r \to \infty$. New equations of motion (for which the infinite distance is a singularity) and the corresponding first integrals of energy and angular momentum are set up. The infinity singularity is blown up via McGehee-type transformations, and the infinity manifold is pasted on the phase space. The fictitious flow on this manifold is described. Then, resorting to the rotational symmetry of the problem and to the angular momentum integral, the near-escape local flow is depicted. The corresponding phase curves are interpreted as physical motions.

1. INTRODUCTION

The study of the zonal satellite problem (the two-body problem associated to potentials of the type $\sum_{k=1}^{n} a_k/r^k$) was started by Mioc and Stavinschi (1998) (hereafter Paper I). They used the McGehee-type transformations of the second kind (McGehee 1974) to blow up the collision singularity t^* ($r = 0$). The first step of these transformations led to the motion equations

$$\begin{aligned} \dot{r} &= \xi, \\ \dot{\theta} &= \eta/r, \\ \dot{\xi} &= \eta^2/r - \sum_{k=1}^{n} k a_k/r^{k+1}, \\ \dot{\eta} &= -\xi\eta/r, \end{aligned} \tag{1}$$

and to the first integrals of energy and angular momentum

$$\xi^2 + \eta^2 - 2\sum_{k=1}^{n} a_k/r^k = h; \tag{2}$$

$$r\eta = C, \tag{3}$$

where (r, θ) are the standard polar coordinates, (ξ, η) stand for the polar components of the velocity, h is the energy constant, while C is the angular momentum constant. (We need equations (1)-(3) for the comments at the end of this paper.)

After the last step of the transformations (see Paper I), equations (1)-(3) became respectively

$$\begin{aligned} r' &= rx, \\ \theta' &= y, \\ x' &= nx^2/2 + y^2 - \sum_{k=1}^{n} k a_k r^{n-k}, \\ y' &= (n/2 - 1)xy; \end{aligned} \tag{4}$$

$$x^2 + y^2 = hr^n + 2\sum_{k=1}^{n} a_k r^{n-k}; \tag{5}$$

$$y^2 = C^2 r^{n-2}. \tag{6}$$

where $' = d/ds$, and s is a timelike variable introduced via $ds = r^{-n/2-1}dt$.

In Paper I it was described the collision manifold M_0 and the flow on it. Then, reducing the 4D full phase space to dimension 2, the local flow near collision was depicted in the phase plane (r, x).

This paper tackles, in a somewhat symmetrical manner, another limit situation: the escape. In Section 2, starting from equations (4)-(6), we establish a new system of motion equations (for which the infinite distance is a singularity), and the corresponding first integrals. Then we perform a set of McGehee-type transformations to blow up the infinity singularity and to replace it by the *infinity manifold* (M_∞) pasted on the phase space.

Section 3 describes the infinity manifold, which is a 2D torus (imbedded in the 4D full phase space) if $h > 0$, a circle if $h = 0$, and the empty set if $h < 0$ (always bounded motion). The flow on M_∞ (consisting of two circles of degenerate equilibria connected by heteroclinic orbits moving from the lower circle to the upper one) has no physical significance, but it provides informations about the local flow near escape (due to the continuity of solutions with respect to the initial data).

The near-escape flow is described in Section 4. As in Paper I, we use successively the rotational symmetry of the global flow and the angular momentum integral to reduce the dimension of the new full phase space from 4 to 2. This allows clear pictures of the local flow near escape for the whole possible interplay between the energy constant h and the field parameter a_1. The local phase curves are interpreted in terms of physical motion, offering an image of the particle behaviour at very great distance from the centre.

2. EQUATIONS OF MOTION AND FIRST INTEGRALS

We start from the system (4). To study the motion for $r \to \infty$, we use the real analytic diffeomorphism

$$\begin{aligned}&[0,\infty)\times[0,2\pi]\times\mathbf{R}\times\mathbf{R}\times[0,\infty)\to(0,\infty)\times\\&\qquad\times[0,2\pi]\times\mathbf{R}\times\mathbf{R}\times[0,s^*)\\&\qquad(r,\theta,x,y;s)\mapsto(\rho,\theta,x,y;s)\end{aligned}$$

defined by

$$\rho = r^{-1}. \tag{7}$$

The motion equations (4) acquire the form

$$\begin{aligned}\rho' &= -\rho x,\\ \theta' &= y,\\ x' &= nx^2/2 + y^2 - \sum_{k=1}^{n} k a_k \rho^{k-n},\\ y' &= (n/2-1)xy,\end{aligned}\tag{8}$$

whereas the first integrals (5) and (6) become respectively

$$\rho^n x^2 + \rho^n y^2 = h + 2\sum_{k=1}^{n} a_k \rho^k; \tag{9}$$

$$y^2\rho^{n-2} = C^2. \tag{10}$$

The above equations present the singularity s^* for $\rho = 0$. To perform the second step, we resort to the real analytic diffeomorphism

$$\begin{aligned}&(0,\infty)\times[0,2\pi]\times\mathbf{R}\times\mathbf{R}\times[0,s^*)\to(0,\infty)\times\\&\qquad\times[0,2\pi]\times\mathbf{R}\times\mathbf{R}\times[0,s^*)\\&\qquad(\rho,\theta,x,y;s)\mapsto(\rho,\theta,u,v;s)\end{aligned}$$

defined by

$$\begin{aligned}u &= \rho^{n/2}x,\\ v &= \rho^{n/2}y.\end{aligned}\tag{11}$$

In this way, equations (8)-(10) become respectively

$$\begin{aligned}\rho' &= -u/\rho^{n/2-1},\\ \theta' &= v/\rho^{n/2},\\ u' &= -(n/2)u^2/\rho^{n/2} - (n/2-1)v^2/\rho^{n/2} +\\ &\quad + (n/2)h/\rho^{n/2} + \sum_{k=1}^{n}(n-k)a_k\rho^{k-n/2},\\ v' &= -uv/\rho^{n/2};\end{aligned}\tag{12}$$

$$u^2 + v^2 = h + 2\sum_{k=1}^{n} a_k\rho^k; \tag{13}$$

$$v = C\rho. \tag{14}$$

The singularity at $\rho = 0$ persists in (12). To blow it up, we introduce a new timelike variable τ via the real analytic diffeomorphism

$$\begin{aligned}&(0,\infty)\times[0,2\pi]\times\mathbf{R}\times\mathbf{R}\times[0,s^*)\to[0,\infty)\times\\&\qquad\times[0,2\pi]\times\mathbf{R}\times\mathbf{R}\times[0,\infty)\\&\qquad(\rho,\theta,u,v;s)\mapsto(\rho,\theta,u,v;\tau)\end{aligned}$$

defined by

$$d\tau = \rho^{-n/2}ds. \tag{15}$$

The vector field (12) becomes

$$\begin{aligned}d\rho/d\tau &= -\rho u,\\ d\theta/d\tau &= v,\\ du/d\tau &= -(n/2)u^2 - (n/2-1)v^2 +\\ &\quad + (n/2)h + \sum_{k=1}^{n}(n-k)a_k\rho^k,\\ dv/d\tau &= -uv,\end{aligned}\tag{16}$$

where we kept by abuse the same notations for the new functions of τ. It is clear that the first integrals (13) and (14) preserve their expressions.

3. INFINITY MANIFOLD

The McGehee-type transformations performed in Section 2 made equations (16) and integrals (13)-(14) well defined for $\rho = 0$. So, the phase space can be analytically extended to contain the manifold

$$M_{\text{inf}} = \{(\rho, \theta, u, v) \mid \rho = 0\},$$

invariant under the flow, because $d\rho/d\tau = 0$ when $\rho = 0$. The integrals (13) and (14) also extend smoothly to the boundary $\rho = 0$.

With (13), we define the constant energy manifold

$$M_h = \{(\rho, \theta, u, v) \mid u^2 + v^2 = h + 2\sum_{k=1}^{n} a_k \rho^k\},$$

corresponding to a fixed energy level. The intersection $M_\infty = M_{\text{inf}} \cap M_h$ provides the so-called *infinity manifold*

$$M_\infty = \{(\rho, \theta, u, v) \mid \rho = 0, \theta \in S^1, u^2 + v^2 = h\}. \tag{17}$$

For $h > 0$, M_∞ is a 2D cylinder (or a 2D torus, since $\theta \in S^1$; see Paper I) imbedded in the 4D full phase space of the McGehee-type coordinates (ρ, θ, u, v) (Figure 1).

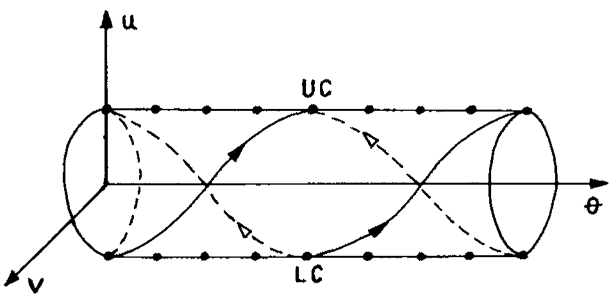

Fig. 1. *The M_∞ cylinder and the flow on it.*

For $h = 0$, M_∞ reduces to the circle $(\theta \in S^1, u = 0, v = 0)$. For $h < 0$, $M_\infty = \emptyset$ (this means that for negative total energy the particle cannot escape; the motion remains always bounded).

As in the case of the collision manifold, it is easy to observe that, by virtue of (15), the M_∞ torus (circle if $h = 0$) is formed only by equilibria of the flow on the full phase space.

The flow on M_∞ has no more physical significance than that on the collision manifold M_0. However, due to the continuity of the solutions with respect to the initial data, it provides valuable informations about the near-escape orbits.

By (13) and (16), the vector field on M_∞ reads

$$\begin{aligned} d\theta/d\tau &= v, \\ du/d\tau &= v^2, \\ dv/d\tau &= -uv. \end{aligned} \tag{18}$$

The critical points of the above equations are $(\theta, u, v) = (\theta_e, \pm\sqrt{h}, 0)$, with arbitrary $\theta \in S^1$. They form two circles of degenerate equilibria on the M_∞ torus: the *upper circle* (UC): $(\theta \in S^1, u = \sqrt{h}, v = 0)$, and the *lower circle* (LC): $(\theta \in S^1, u = -\sqrt{h}, v = 0)$. Because, by (18), $du/d\tau > 0$ for $v \neq 0$, all other phase curves on M_∞ are heteroclinic and move from LC to UC. Analogous to Paper I, we put $u = \sqrt{h}\cos\alpha$, $v = \sqrt{h}\sin\alpha$, obtaining from (18) $d\alpha/d\theta = -1$. The flow on M_∞ is illustrated in Figure 1.

Having in view (15), we see that all orbits which approach infinity neighbour M_∞ in the zone of UC (the escaping ones) and LC (those which come from infinity).

Observe that for $h = 0$ the infinity manifold reduces to a circle of degenerate equilibria (UC and LC identified).

Two more characteristics of such a motion must be emphasized. First, the infinity manifold does exist only for $h \geq 0$. In other words, the motion with negative total energy remains always bounded. Second, taking into account (13), the asymptotic velocity of the particle at infinity is positive for $h > 0$ and zero for $h = 0$.

4. NEAR-ESCAPE FLOW

To describe the local flow near the infinity manifold, we act as in case of the collision manifold (Paper I). Using the rotational symmetry of the problem (θ does not appear explicitly in (16) and (13)), we factorize the flow to S^1, obtaining the 3D reduced phase space (RPS) of the coordinates (ρ, u, v). Next, we reduce RPS to the 2D phase plane PP of the coordinates (ρ, u) by using (14). In this way the energy relation (13) becomes in PP:

$$u^2 = \sum_{j=o}^{n} A_j \rho^j, \tag{19}$$

with $A_0 = h$, $A_1 = 2a_1$, $A_2 = 2a_2 - C^2$, $A_j = 2a_j$ $(j = \overline{3, n})$. We shall consider very small values of ρ (the terms in ρ^i, $i \geq 2$, will be neglected).

If $h > 0$, the M_∞ torus in full phase space becomes in RPS the circle $\{\rho = 0, u^2 + v^2 = \sqrt{h}\}$, which, in its turn, reduces in PP to the points M $(\rho = 0, u = +\sqrt{h})$ and N $(\rho = 0, u = -\sqrt{h})$. If $h = 0$, the M_∞ circle becomes the origin $(\rho = 0, u = 0, v = 0)$ in RPS, and the origin $(\rho = 0, u = 0)$ in PP, too.

The local structure of the phase space is depicted in Figure 2. Figure 2a plots the phase trajectories for $h > 0$. If $a_1 < 0$, we have the curves 1: heteroclinic trajectories which eject from M and then tend to N. This physically means motion coming from infinity, moving inwards up to a minimum distance $r_m = -2a_1/h$ (corresponding to $\rho = -h/(2a_1)$), then tending back to infinity. If $a_1 = 0$, the phase portrait consists of the two halflines 2 and

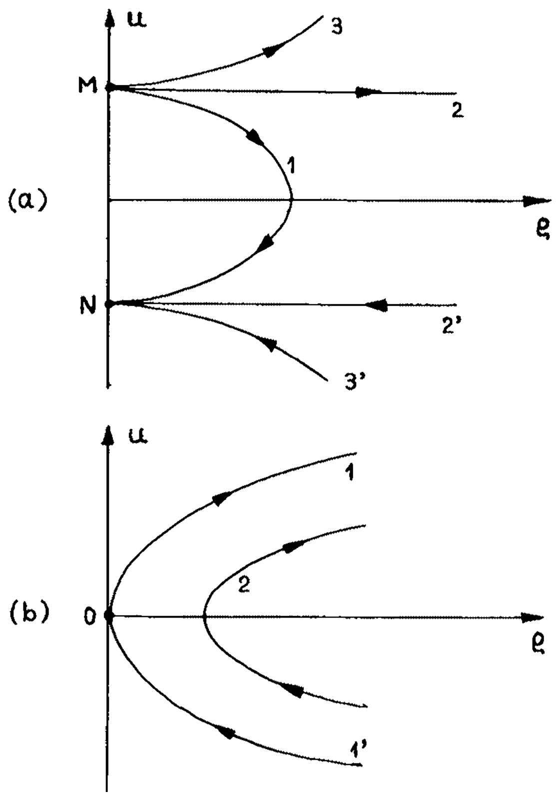

Fig. 2. *The near-infinity flow in PP for: (a) $h > 0$; (b) $h \leq 0$.*

2' (coming from infinity, or tending to infinity, respectively). If $a_1 > 0$, we have the curves 3 and 3', with similar significances as the curves 2 and 2', respectively.

The case $h = 0$ is illustrated in Figure 2b. If $a_1 < 0$, the real motion is impossible (in our first approximation with local character). If $a_1 = 0$, the phase portrait is represented by the origin ($\rho = 0, u = 0$): degenerate orbit remaining forever at infinity (also in this first approximation). If $a_1 > 0$, we have the curves 1 and 1', which physically represent motion coming from infinity, or tending to infinity, respectively.

Lastly, consider $h < 0$. The real motion is impossible for $a_1 \leq 0$ (in our local approximation). For $a_1 > 0$, the phase portrait is given in Figure 2b, too (curves 2). The particle moves away from the centre, up to a maximum distance $r_M = -2a_1/h$ (corresponding to $\rho = -h/(2a_1)$), then returns towards the centre: no possibility of escape.

It is needless to say that the character of these physical motions at very great distances from the centre is spiral for $C \neq 0$, and radial for $C = 0$.

5. COMMENTS

A first remark concerns the manner of obtaining the equations of motion (16). It is clear from (15) and from the connection between s and t (see Section 1) that $dt = rd\tau$ (which reveals, on the other hand, that the flows on M_0 and M_∞ have different time scales). On this basis, and using (1), one easily remarks that (u, v) are nothing but the polar components of the velocity (ξ, η). All that being established, equations (16) follow straightforwardly from the system (1). So, the start from the regularized equations (4) could be avoided; nevertheless, we preferred this way which seemed more elegant to us.

The same comments are valid as regards the first integrals (13) and (14), which, by virtue of the above specifications, could directly be deduced from equations (5) and (6), respectively.

Another fact deserves to be mentioned: in order to obtain the vector field (16), intended here to study the infinity manifold, the fictitious flow on it, and the local flow near escape, we need neither start from already regularized equations as (4), nor use equation-regularizing transformations as those of McGehee-type are. There is no serious reason for choosing the technique used in Section 2; the only argument is that pointed out above.

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ПРОБЛЕМ ЗОНСКИХ САТЕЛИТА. II. ПРОТОК У ОБЛАСТИ БЛИЗУ КРИТИЧНЕ

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Наставља се проучавање проблема зонских сателита разматрањем ситуације $r \to \infty$. Изведене су нове једначине кретања (где се бесконачна удаљеност појављује као сингуларност) и одговарајући први интеграли енергије и момента импулса. Сингуларност услед бесконачности се одстрањује уз помоћ трансформација МекГеeовог типа и скуп бесконачности се пресликава на фазни простор. Описан је фиктивни проток на овом скупу. Затим се описује локални проток у области близу критичне позивањем на обртну симетрију проблема и интеграл момента импулса. Одговарајуће фазне криве се интерпретирају као физичка кретања.