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Alternative Representations of the Radiant Energy Transport

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The Stellar Atmosphere Physical System

Two components:

- ***radiation field***
- ***matter***

Representation of the physical system:

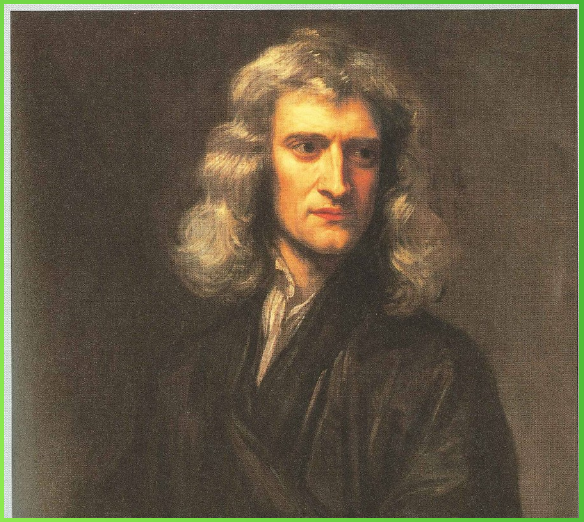
macroscopic ***and*** ***microscopic*** ***description***

- *The physical ground of* ***radiative transfer*** *is the propagation of radiation through a medium, namely a* ***transport process***

- *Any transport process is* ***characterized by*** *the*
flux of a proper quantity

- *We will consider the* ***specific*** *transport process where* ***radiant energy*** *is carried on through a* ***medium*** *with which it* ***exchanges energy****.*

Ab initio



God said, Let Newton be!
And all was light
(A. Pope)

Rays of light

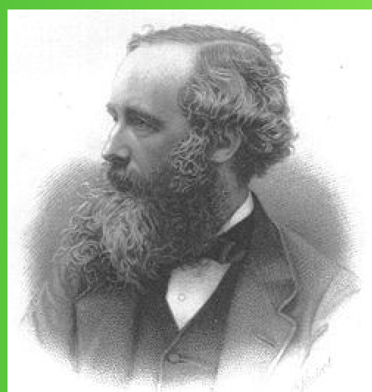
The least Light, or part of Light, which may be stopp'd alone without the rest of the Light, or propagated alone, or do suffer anything alone, which the rest of the Light doth not or suffers not, I call a Ray of Light.
(Optiks, 1704)

Light is made of corpuscle

The dual nature of light

Huygens wave theory (17th century)

confirmed experimentally by Young and Fresnel

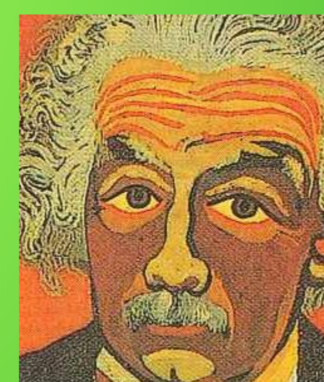


Maxwell's electromagnetic waves

revealed by Hertz's experiments

Einstein's hypothesis of the

quantum of light



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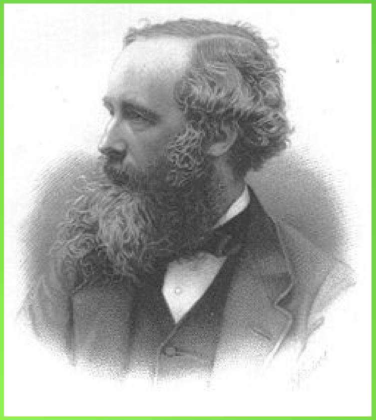
6. ***Transport like a fluid dynamics process;***
7. ***Comparison between the electrodynamic and the macroscopic picture.***

1. Electrodynamic formulation

James Clark Maxwell
(1831 - 1879)



A Dynamical Theory of the Electromagnetic Field (1865)

$$\nabla \cdot \boldsymbol{D} = 4\pi\rho \quad ;$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad ;$$

$$\nabla \times \boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = 0 \quad ;$$

$$\nabla \times \boldsymbol{H} + \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} = \frac{4\pi}{c}\boldsymbol{J} \quad .$$

War es ein Gott,
Der diese Zeichen
schreib?

Following Maxwell:

magnetic and electric ***energy density***

$$W_{mag} = \frac{1}{8\pi}\boldsymbol{H}\cdot\boldsymbol{B} \quad ; \qquad W_{elec} = \frac{1}{8\pi}\boldsymbol{D}\cdot\boldsymbol{E} \quad .$$

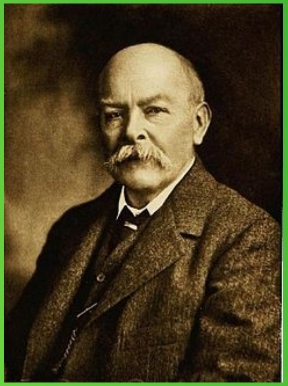
*Energy is **localized** in the field.*

We adopt the Gauss conventional system of units, where

$$[E] = [D] = [B] = [H] = M^{1/2} L^{-1/2} T^{-1}$$

$$\varepsilon = \mu = 1 \quad ; \qquad [\varepsilon] = [\mu] = M^0 L^0 T^0$$

John H. Poynting
(1852 – 19 14)



Poynting's vector: $\boldsymbol{S} \equiv \frac{c}{4\,\pi}\,\boldsymbol{E} \times \boldsymbol{H}$

$$[S] = \left(L\,T^{-1}\right)\left(M^{1/2}\,L^{-1/2}\,T^{-1}\right)^2 = \left(M\,L^2\,T^{-2}\right)T^{-1}\,L^{-2}$$

i.e. $\frac{energy}{time \cdot surface} = power\ flux$

By a proper treatment of the last two Maxwell's equations

$$\rightarrow \quad \frac{1}{4\pi}\,\boldsymbol{H}\cdot\dot{\boldsymbol{B}} + \frac{1}{4\pi}\,\boldsymbol{E}\cdot\dot{\boldsymbol{D}} + \boldsymbol{E}\cdot\boldsymbol{J} + \nabla\boldsymbol{S} = 0$$

Poynting's theorem

$$[each\ term] = M\,L^{-1}T^{-3} = \left(M\,L^2T^{-2}\right)T^{-1}\,L^{-3}$$ *i.e. power density*

Joule heat: $W_J \equiv \boldsymbol{E}\cdot\boldsymbol{J}$

$$W \equiv W_{elec} + W_{mag}$$

It can be shown that $\dot{W}_{elec} = \frac{1}{8\pi}\boldsymbol{E}\cdot\dot{\boldsymbol{D}} + \frac{1}{8\pi}\dot{\boldsymbol{E}}\cdot\boldsymbol{D} = \frac{1}{4\pi}\boldsymbol{E}\,\dot{\boldsymbol{D}}$

The same for $\dot{W}_{mag}$.

Hence from Poynting's theorem

$$\dot{W} + \nabla\cdot\boldsymbol{S} = -W_J \ .$$

energy balance of the electromagnetic field

By integration over a volume V and Gauss theorem

$$\int_\Sigma \boldsymbol{S}\cdot\boldsymbol{n}\,d\sigma = -\int_V \left[\frac{\partial W}{\partial t} + W_J\right] dV$$

conservation equation

Physical meaning of the Poynting's vector:

energy flux per unit time
across ***unit area*** *of the*
boundary surface of the volume considered

→ ***Transport of energy of the electromagnetic field***

The Poynting's ***vector*** *accounts for the*
intrinsic directed aspect
of the ***propagation of the electromagnetic field****.*

Transport of radiant energy by an e.m. wave

monochromatic polarized plane wave

propagating along the x – axis, specified by $\hat{\boldsymbol{x}}$

$\boldsymbol{E} \perp \boldsymbol{H}$: *only* E_y *and* H_z *not* 0

$$\frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} - \frac{\partial^2 E_y}{\partial x^2} = 0 \qquad \textit{wave equation}$$

solution $E_y(x,t) = E_0 \cos(k x - \omega t)$

by proper manipulation

$$\frac{\partial}{\partial t}\left\{ \frac{1}{8\pi k^2}\left[\frac{1}{c^2}\left(\frac{\partial E_y}{\partial t}\right)^2 + \left(\frac{\partial E_y}{\partial x}\right)^2 \right] \right\} - \frac{\partial}{\partial x}\left(\frac{1}{4\pi k^2}\frac{\partial E_y}{\partial t}\frac{\partial E_y}{\partial x} \right) = 0 \ .$$

$$e \equiv \frac{1}{8\pi k^2}\left[\frac{1}{c^2}\left(\frac{\partial E_y}{\partial t}\right)^2 + \left(\frac{\partial E_y}{\partial x}\right)^2 \right] \qquad f \equiv -\left(\frac{1}{4\pi k^2}\frac{\partial E_y}{\partial t}\frac{\partial E_y}{\partial x} \right)$$

From the previuos definitions:

$$\frac{\partial e}{\partial t} + \frac{\partial f}{\partial x} = 0 \ .$$

wave equation => ***equation of continuity***

$$[e] = \left(M L^2 T^{-2}\right) L^{-3} \ ; \qquad [f] = \left(M L^2 T^{-2}\right) T^{-1} L^{-2}$$

energy density ***power flux***

$$e(t) = \frac{E_0^{\ 2}}{4\pi} \sin^2(kx - \omega t)$$

$$W(t) = W_{elec}(t) + W_{mag}(t) = \frac{E_0^{\ 2}}{4\pi} \cos^2(kx - \omega t)$$

$$e \iff W$$

$$f(t) = \frac{E_0^{\ 2}}{4\pi} \frac{\omega}{k} \sin^2(kx - \omega t) = \frac{c}{4\pi} E_0^{\ 2} \sin^2(kx - \omega t)$$

$$\boldsymbol{S}(t) = \frac{c}{4\pi} E_y^2(t)\,\hat{\boldsymbol{x}} = \frac{c}{4\pi} E_0^{\ 2} \cos^2(kx - \omega t)\,\hat{\boldsymbol{x}} \ .$$

$$f\ \hat{\boldsymbol{x}} \iff \boldsymbol{S}$$

2. Fluid dynamic – like picture

Picture based on macroscopic quantities

related to the microscopic photon picture

(corpuscular model of the radiation field)

Analogue with fluid dynamics:

macroscopic flux of particles propagating along the

***paths of geometrical optics* (eikonal equation)**

that carry on and exchange energy with matter particles

Ray :

amount of radiant energy of frequency ν

carried on along the direction $\mathbf{n}$ *with speed* c

per unit time

across a unit surface perpendicular to $\mathbf{n}$

rays $\Longleftrightarrow$ *transport of energy*

Under the assumptions of

a weak electromagnetic field and
propagation through a diluted medium

the energy carried on by rays obeys the empirical laws of

radiometry

1. ***propagation through vacuum along straight lines with speed c;***
2. ***all rays through a given point are independent;***
3. ***they are linearly additive both in direction and frequency.***

(Hypotheses already formulated by Newton in his Opticks)

The above laws of photometry warrants that

the transport process is intrinsically linear

However

a ***single directed*** *quantity (i.e. a vector)* ***is not enough*** *to specify* ***completely*** *the radiation firld:*

virtually infinite pencil of rays

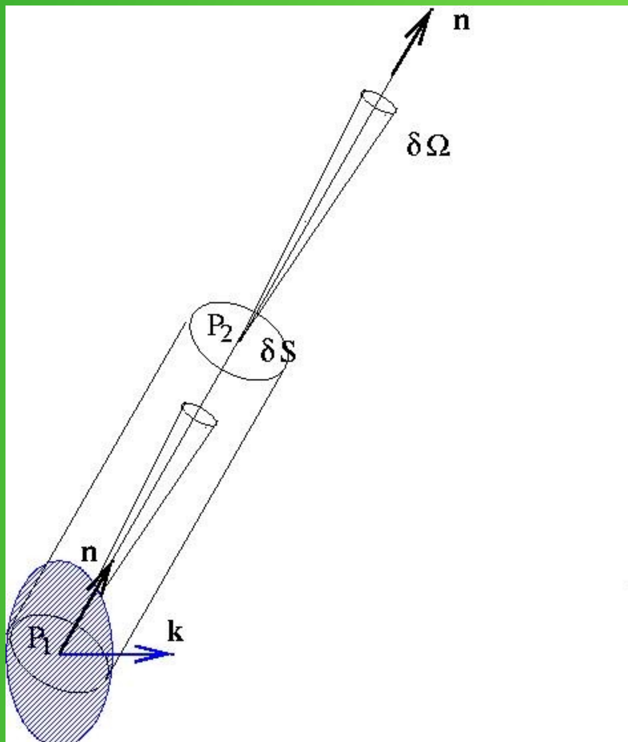
From rays to specific intensity

Fundamental physical observable in radiative tyranfer:

the energy carried on by a ray

→ ***Scalar*** *macroscopic* ***local*** *and* ***directed*** *quantity :*

specific intensity of the radiation field



observable:

amount of energy $\delta E_\nu(\boldsymbol{n})$

elements of the measure:

oriented surface $\boldsymbol{k}\,\delta S$ *around* P_1

solid angle $\delta\Omega$ *around* $\boldsymbol{n}$

time interval δt *spectral range* $\delta\nu$

$$\delta E_\nu(\boldsymbol{n}) \propto \boldsymbol{n}\cdot\boldsymbol{k}\,\delta S\;\delta\Omega\;\delta\nu\;\delta t$$

$$(\boldsymbol{n}\cdot\boldsymbol{k})^{-1}\lim_{\delta S\,\delta\Omega\,\delta\nu\,\delta t\to 0}\frac{\delta E_\nu(\boldsymbol{n})}{\delta S\;\delta\Omega\;\delta\nu\;\delta t} \equiv I(\boldsymbol{r},t;\boldsymbol{n},\nu)$$

By definition the specific intensity $I\left(\boldsymbol{r}, t; \boldsymbol{n}, \nu\right)$

characteyerized by $\left(\boldsymbol{n}, \nu\right)$

is the coefficient of proportionality

between the

observable and the elements of the measurement

Dimension:

$$[I] = \left(M L^2 T^{-2}\right) \cdot L^{-2} \cdot T^{-1} \cdot T$$

i.e. ***energy flux per unit time and unit frequency band***

Moments of the specific intensity

0 th order moment: ***average mean intensity***

$$J(\boldsymbol{r},t;\nu) \equiv \frac{1}{4\pi}\oint I(\boldsymbol{r},t;\boldsymbol{n},\nu)\, d\boldsymbol{n}$$

scalar

1 st order moment: ***flux of radiation***

$$\boldsymbol{F}_{\nu}(\boldsymbol{r},t) \equiv \oint I(\boldsymbol{r},t;\boldsymbol{n},\nu)\, \boldsymbol{n}\, d\boldsymbol{n}$$

vector

2 nd order moment: ***radiation pressure***

$$\underline{\underline{\boldsymbol{T}_{\nu}}}(\boldsymbol{r},t) \equiv \frac{1}{c}\oint I(\boldsymbol{r},t;\boldsymbol{n},\nu)\, \boldsymbol{n}\boldsymbol{n}\, d\boldsymbol{n}$$

tensor

Dyadic notation

Energy density of the radiation field

In the time interval dt ***the volume*** $dV = \boldsymbol{n}\cdot\boldsymbol{k}\, dS\, c\, dt$ ***is filled in by radiant energy***

Specific energy density: $U(\boldsymbol{r}, t; \boldsymbol{n}, \nu) \equiv \frac{d\, E_\nu(\boldsymbol{n})}{dV}$

directed *and* ***spectral***

By definition $U(\boldsymbol{r}, t; \boldsymbol{n}, \nu)\, d\Omega\, d\nu = \frac{1}{c}\, I(\boldsymbol{r}, t; \boldsymbol{n}, \nu)\, d\Omega\, d\nu$

By integration over all the directions

$$u_\nu \equiv u(\boldsymbol{r}, t; \nu) \equiv \frac{1}{c} \oint I(\boldsymbol{r}, t; \boldsymbol{n}, \nu)\, d\Omega = \frac{4\pi}{c}\, J(\boldsymbol{r}, t; \nu)$$

→ ***spectral***

$$[u_\nu] = (M\, L^2\, T^{-2})\, L^{-3}\, T$$

3. Microscopic picture

The photon distribution function

Because of the ***corpuscular nature*** *of photons, let us define a* ***distribution function*** ***such that***

$$f(\mathbf{r}, t; \mathbf{n}, \nu)\, d\Omega\, d\nu$$

is ***equal*** *to the*

nr. of photons per unit volume *at* **r** *and* ***t***

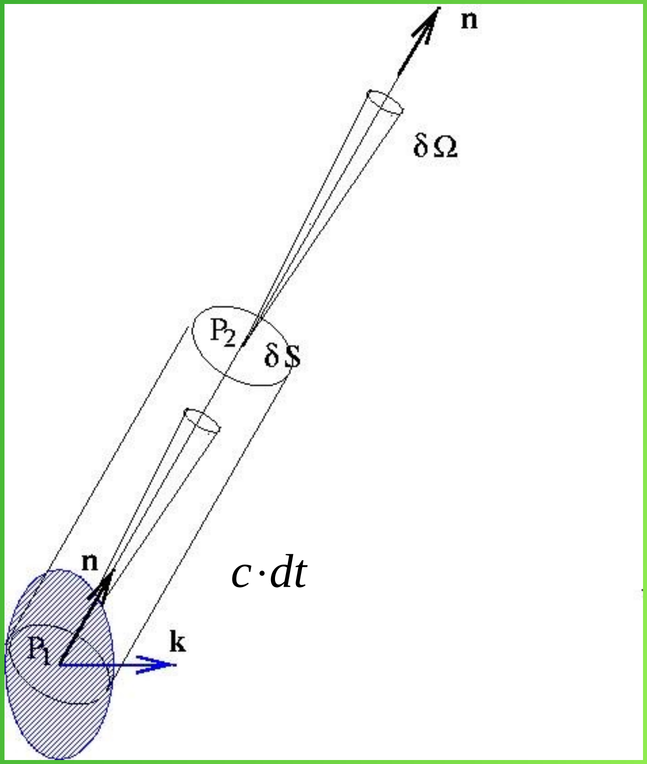
in the band $(\nu, \nu + d\nu)$

that propagates along **n** *with speed c into* $d\Omega$.

f is characterized by the pair $(\mathbf{n}; \nu)$

directed *and* ***spectral***

$$[f] = L^{-3} \cdot T$$



The **number of** ***specific*** **photons**

crossing the surface $\boldsymbol{k}\cdot\boldsymbol{n}\,dS$ *into* $d\Omega$

during $\boldsymbol{dt}$ *to fill a volume* $dV = \boldsymbol{n}\cdot\boldsymbol{k}\,dS\;c\;dt$

is by definition

$$f\left(\boldsymbol{r},t\,;\boldsymbol{n},\nu\right)\,\boldsymbol{n}\cdot\boldsymbol{k}\,dS\;c\;dt\;\;d\Omega\;d\nu$$

Each photon carries on its energy $h\nu$

Transport process in terms of the photon distribution function

Specific energy ***flowing through*** $\boldsymbol{k}\cdot\boldsymbol{n}\,dS$

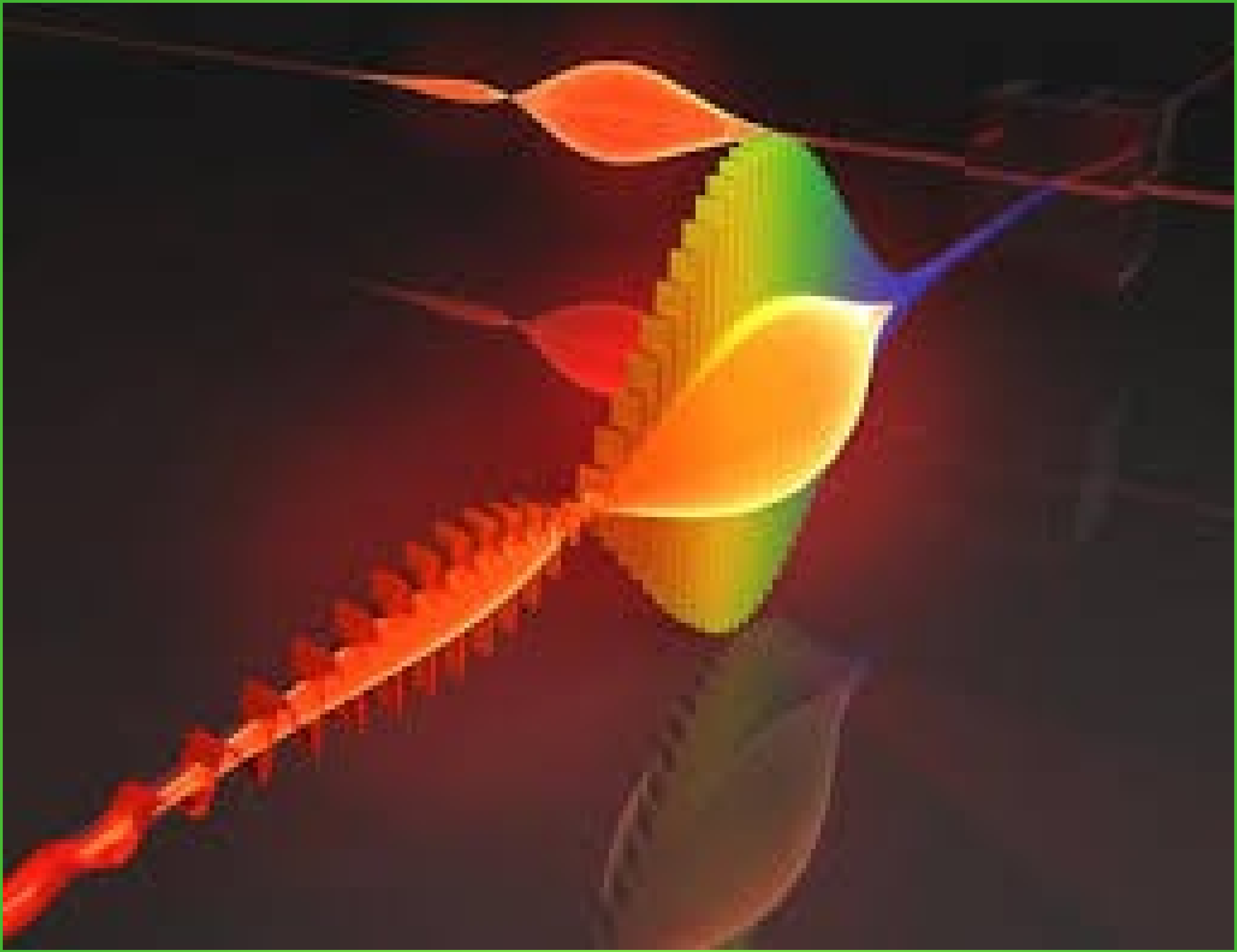
$$d\,E_\nu\left(\boldsymbol{n}\right)\;=\;h\nu\;c\;f\left(\boldsymbol{r},t\,;\boldsymbol{n},\nu\right)\,\boldsymbol{n}\cdot\boldsymbol{k}\,dS\;d\Omega\;d\nu\,dt$$

By direct comparison

$$I\left(\boldsymbol{r},t\,;\boldsymbol{n},\nu\right)\;=\;c\,h\nu\;f\left(\boldsymbol{r},t\,;\boldsymbol{n},\nu\right)$$

Time for a cup of tea

(May be a pint would be better)



4. The RT equation as a kinetic equation for photons

From a ***formal standpoint***

a ***kinetic equation*** *for any* ***transported quantity*** *is formally*

Total rate of change = Source terms – Sink terms

(Boltzmann's equation)

Total rate of change = Eulerian derivative:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}} + \dot{\boldsymbol{p}} \cdot \frac{\partial}{\partial \boldsymbol{p}}$$

In our case

Sources and sinks determined by:

atomic properties of the interaction matter - radiation

equation of state of matter (LTE) or SE equations

Distribution function $F(\boldsymbol{r},\boldsymbol{p},t)$ ***for photons with momentum***

$$\boldsymbol{p} = \boldsymbol{n}\,\frac{h\nu}{c} \quad ; \qquad \boldsymbol{p} = \boldsymbol{p}(\boldsymbol{n},\nu)$$

Kinetic equation:

$$\frac{d}{dt}F(\boldsymbol{r},\boldsymbol{p},t) = \left[\frac{\delta F}{\delta t}\right]_{sources} - \left[\frac{\delta F}{\delta t}\right]_{sinks}$$

It can be shown that

$$f(\boldsymbol{r},t;\boldsymbol{n},\nu) = \frac{h^3\nu^2}{c^3}F(\boldsymbol{r},\boldsymbol{p},t)$$

$$I(\boldsymbol{r},t;\boldsymbol{n},\nu) = c\,h\nu\, f(\boldsymbol{r},t;\boldsymbol{n},\nu)$$ ***parameters*** $(\boldsymbol{n},\nu)$

$$\rightarrow \qquad F(\boldsymbol{r},\boldsymbol{p},t) = \frac{c^2}{h^4\nu^3}I(\boldsymbol{r},t:\boldsymbol{n},\nu)$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \dot{\boldsymbol{p}}\cdot\frac{\partial}{\partial \boldsymbol{p}} \quad ; \qquad \frac{\partial}{\partial \boldsymbol{r}} = \nabla$$

For photons $\boldsymbol{v} = c\,\boldsymbol{n}$ *and* $\dot{\boldsymbol{p}} = 0$

$$\rightarrow \quad \frac{1}{c}\frac{\partial}{\partial t} I(\boldsymbol{r},t:\boldsymbol{n},\nu) + \boldsymbol{n}\cdot\nabla\, I(\boldsymbol{r},t;\boldsymbol{n},\nu) =$$

$$\frac{1}{c}\left[\frac{\delta I}{\delta t}\right]_{sources} - \frac{1}{c}\left[\frac{\delta I}{\delta t}\right]_{sinks} = \left[\frac{\delta I}{\delta l}\right]_{sources} - \left[\frac{\delta I}{\delta l}\right]_{sinks}$$

where $\delta l = c\,\delta t$ *is a* ***path length*** *along* $\boldsymbol{n}$

Radiative Transfer equation:

$$\frac{1}{c}\frac{\partial}{\partial t} I(\boldsymbol{r},t:\boldsymbol{n},\nu) + \boldsymbol{n}\cdot\nabla\, I(\boldsymbol{r},t;\boldsymbol{n},\nu) = \left[\frac{\delta I}{\delta l}\right]_{sources} - \left[\frac{\delta I}{\delta l}\right]_{sinks}$$

mathematical formulation of a directional problem

in terms of the macroscopic quantity specific intensity

For any specific intensity ,
characterized by the pair of parameters $(\boldsymbol{n};\nu)$

one specific RT equation

Each term in the RT equation has dimension

$$\left(M\,L^{2}T^{2}\right)L^{-2}\,L^{-1} = M\,L^{-1}\,T^{-2}$$

From a linear to non-linear problem

Mathematical complications arise when

the individual specific RT equations are coupled together

through the Source and Sink terms

Non-local problems

Moreover the transport process necessarily implies

non-local effects

brought about by matter-radiation interaction

5. Macroscopic RT coefficients

Consistently with the macroscopic picture

we consider ***homogeneous volume elements*** *that* ***emit*** *and* ***absorb*** *radiant energy* ***isotropically***

All the physical information *at* ***atomic level*** *is incorporated into a* ***limited number*** *of* ***macroscopic coefficients***

Thermal emission coefficient

ΔE_ν^{th} *energy emitted along* **n**

by ΔV

into $\Delta \Omega$

during Δt

in $(\nu, \nu + \Delta \nu)$

measurable quantity

parameters of the measure

$$\Delta E_\nu^{th} \propto \Delta V \; \Delta \Omega \; \Delta \nu \; \Delta t$$

$$\lim_{\Delta\sigma \; \Delta\Omega \; \Delta\nu \; \Delta t \to 0} \frac{\delta E_\nu^{th}}{\delta V \; \delta \Omega \; \delta \nu \; \delta t} \equiv \eta_\nu^{th}$$

Decrease of the specific intensity

along a path δl *in the direction* **n**

True absorption coefficient $a_\nu(\boldsymbol{n})$:

fraction of energy ***removed***

converted into internal energy

$$\delta I(\boldsymbol{n}) \propto I(\boldsymbol{n})\,\delta l \qquad\qquad \frac{\delta I(\boldsymbol{n})}{I(\boldsymbol{n})} = a_\nu(\boldsymbol{n})\,\delta l$$

Likewise

Scattering coefficient $\sigma_\nu(\boldsymbol{n})$:

fraction of energy ***removed***

diverted into a different direction

Extinction coefficient

Global effect of the attenuation,

i.e., removal of photons from a given beam

$$\chi_\nu(\boldsymbol{n}) \equiv a_\nu(\boldsymbol{n}) + \sigma_\nu(\boldsymbol{n})$$

$$[\chi_\nu] = [a_\nu] = [\sigma_\nu] = L^{-1}$$

Factorization of the macroscopic coefficients

$$coefficient \;=\; cross\ section \;\times\; nr.of\ carriers$$

$$a(\nu) \;=\; a_P(\nu)\, n_{Pa}\,; \qquad \sigma(\nu) \;=\; \sigma_P(\nu)\, n_{Ps}$$

a_P *and* σ_P *atomic data* $\qquad [a_P] = [\sigma_P] = L^2$

n_{Pa} *and* n_{Ps} *populations density* $\qquad [n_{P_a}] = [n_{P_s}] = L^{-3}$

6. Transport like a fluid dynamics process

Analogy between fluid dynamics and radiative transfer

Fluid dynamics considers the motion of ***fluid elements*** *along* ***streamlines***

Macroscopic representation of the radiation field:

The ***amount of specific energy*** $\Delta E_\nu(\boldsymbol{n})$ *carried on along* $\boldsymbol{n}$ *takes the place of the* ***fluid elements***

Correspondence between the ***equations of fluid dynamics*** *and the* ***eikonal equation*** *of* **geometrical optics**

streamlines **<=>** ***rays***

generic ***scalar*** *quantity* $\quad Q = q\, N_c$

q *quantity for an individual particle*

N_c *Nr. of carriers along* $\boldsymbol{n}$

velocity $\quad \boldsymbol{v} = v\,\boldsymbol{n}$

associated *with the* ***vector*** *quantity* $\quad Q\,\boldsymbol{v}$

Space – time evolution of $\quad Q(\boldsymbol{r},t)$:

$$\frac{dQ(\boldsymbol{r},t)}{dt} = \frac{\partial Q(\boldsymbol{r},t)}{\partial t} + \nabla Q(\boldsymbol{r},t)\cdot c\,\boldsymbol{v} = \frac{\partial Q(\boldsymbol{r},t)}{\partial t} + \nabla\cdot\left[Q(\boldsymbol{r},t)\,\boldsymbol{v}\right] .$$

if v constant

If Q is ***conserved*** $\dfrac{dQ(\mathbf{r},t)}{dt} = 0$

=> $\dfrac{\partial Q(\mathbf{r},t)}{\partial t} = -\nabla\cdot\left[Q(\mathbf{r},t)\,\mathbf{v}\right]$ ***continuity equation***

$Q\,v\,(\mathbf{n}\cdot\mathbf{k})\,d\sigma$ *is the* ***flux*** *of* $Q\,\mathbf{v}$ *through* $\mathbf{k}\,d\sigma$

divergence theorem ;

$$\iiint_V \frac{\partial Q(\mathbf{r},t)}{\partial t}\,dV = -\iint_\Sigma Q(\mathbf{r},t)\,v\,(\mathbf{n}\cdot\mathbf{k})\,d\sigma \quad .$$

The integral over some volume V of the ***time derivative***

of the ***transported scalar quantity*** *is equal to the*

flux of the associated vector quantity

through the boundary of V.

In the case of radiative transfer

carriers: ***specific photons*** $\left(\boldsymbol{n}, \nu\right)$

with ***individual energy*** $h\nu$

and ***momentum*** $\frac{h\nu}{c}\,\boldsymbol{n}$

travelling along $\boldsymbol{n}$ *with velocity* $c\,\boldsymbol{n}$

N_c *is given by the photon distribution function*

7. Electrodynamical vs. macroscopical picture

***Correspondence** between*
*the **specific intensity** and the **electric field strength***

***Monochromatic plane wave** of frequency* $\nu_0 = 1/T$

propagating along $\boldsymbol{n}_0 = \boldsymbol{n}_0(\theta_0, \phi_0)$

$$\boldsymbol{n}_0 \equiv \hat{x} \; ; \qquad \hat{x} \perp \boldsymbol{E} \perp \boldsymbol{H}$$

$$[\boldsymbol{E}] = [\boldsymbol{D}] = [\boldsymbol{B}] = [\boldsymbol{H}]$$

The solution of the wave equation

$$\frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} - \frac{\partial^2 E_y}{\partial x^2} = 0$$

is $$E_y(x,t) = E_0 \cos(kx - \omega t)$$

From the average over T of $$W_{elec} \equiv \frac{1}{8\pi}\,\boldsymbol{E}\cdot\boldsymbol{D}$$

and $$W_{mag} \equiv \frac{1}{8\pi}\,\boldsymbol{H}\cdot\boldsymbol{B}\ .$$

=> $$\langle W(t)\rangle_T = \frac{E_0^2}{8\,\pi}\ .$$

Corresponding specific intensity :

$$I\left(\vartheta,\phi,\nu\right)=I_0\,\delta\left(\vartheta-\vartheta_0\right)\delta\left(\phi-\phi_0\right)\delta\left(\nu-\nu_0\right)$$

$$[I]\;=\;M\,T^{-2}\;;\quad[\delta(\nu-\nu_0)]\;=\;T\;;\quad[I_0]\;=\;M\,T^{-3}$$

From the physical standpoint

$$\langle W(t)\rangle_T\;=\;u(\boldsymbol{r},t)$$

$$u_\nu\;\equiv\;u(\boldsymbol{r},t;\nu)\;\equiv\;\frac{1}{c}\oint I(\boldsymbol{r},t;\boldsymbol{n},\nu)\,d\Omega$$

=> $$I_0\;=\;\frac{c}{8\,\pi}\,E_0^2\;,$$

$$[I_0]\;=\;[c\,E_0^2]\;=\;M\,T^{-3}$$

Electromagnetic counter part of $\boldsymbol{F}_{\boldsymbol{\nu}}(\boldsymbol{r},t)$

$$\boldsymbol{F}_{\boldsymbol{\nu}}(\boldsymbol{r},t) \equiv \oint I(\boldsymbol{r},t;\boldsymbol{n},\nu)\,\boldsymbol{n}\,d\boldsymbol{n}$$

is the ***monochromatic power flux*** *of the radiation field*

$$E_y(x,t) = E_0 \cos(kx - \omega t)\,; \qquad \hat{x} \perp \boldsymbol{E} \perp \boldsymbol{H}\,; \qquad |E_0| = |H_0|$$

$$\int_0^\infty d\nu \oint d\boldsymbol{n}\; I(\boldsymbol{r},t;\boldsymbol{n},\nu)\,\boldsymbol{n} = I_0\,\boldsymbol{n_0} = \frac{c}{8\pi} E_0^2\,\boldsymbol{n_0}$$

bolometric vector flux

$$\langle \boldsymbol{S}(t) \rangle_T = \frac{c}{8\pi} E_0^2\,\boldsymbol{n_0}$$

Correspondence of the ***radiative pressure*** *with the*

Maxwell stress tensor

$$\boldsymbol{p}(\boldsymbol{n},\nu) = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \frac{h\nu}{c}\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$$

moment *carried on by a photon* $(\boldsymbol{n},\nu)$

$$\underline{\underline{\boldsymbol{T}_{\boldsymbol{\nu}}}}(\boldsymbol{r},t) \equiv \frac{1}{c}\oint I(\boldsymbol{r},t;\boldsymbol{n},\nu)\,\boldsymbol{n}\boldsymbol{n}\,d\boldsymbol{n}$$

radiative pressure tensor

$$\left[\underline{\underline{\boldsymbol{T}_{\boldsymbol{\nu}}}}\right] = \left(M\,L\,T^{-1}\right)L^{-2}$$ ***flux of momentum***

$$\boldsymbol{k}\cdot\underline{\underline{\boldsymbol{T}_{\boldsymbol{\nu}}}} = \begin{pmatrix} \dfrac{1}{c}\oint I(\boldsymbol{r},t;\boldsymbol{n},\nu)\,n_x\,(\boldsymbol{k}\cdot\boldsymbol{n})\,d\boldsymbol{n} \\ \dfrac{1}{c}\oint I(\boldsymbol{r},t;\boldsymbol{n},\nu)\,n_y\,(\boldsymbol{k}\cdot\boldsymbol{n})\,d\boldsymbol{n} \\ \dfrac{1}{c}\oint I(\boldsymbol{r},t;\boldsymbol{n},\nu)\,n_z\,(\boldsymbol{k}\cdot\boldsymbol{n})\,d\boldsymbol{n} \end{pmatrix}$$

net flux *of* $\boldsymbol{p}_j$ *across unit area* $\boldsymbol{k}$ *:*

$$\oint \frac{1}{c}\,I(\boldsymbol{r},t;\boldsymbol{n},\nu)\,n_j\,(\boldsymbol{k}\cdot\boldsymbol{n})\,d\boldsymbol{n} = \left(\boldsymbol{k}\cdot\underline{\underline{T_{\nu}}}\right)_j$$

$$\left(\boldsymbol{n}\cdot\underline{\underline{\boldsymbol{T}_{\nu}}}\right)_j = \frac{1}{c}\oint I\left(\boldsymbol{r},t;\boldsymbol{n},\nu\right) n_j\, d\boldsymbol{n} = \frac{1}{c}\left(\boldsymbol{F}_{\nu}\right)_j$$

net transport *of* $\boldsymbol{p}$:

$$\boldsymbol{n}\cdot\frac{1}{c}\boldsymbol{F}_{\nu} = \boldsymbol{n}\cdot\oint\frac{h\nu}{c} f\left(\boldsymbol{r},t;\boldsymbol{n},\nu\right) c\boldsymbol{n}\, d\boldsymbol{n}$$

Let us define

$$\boldsymbol{G}_{\nu}\left(\boldsymbol{r},t\right) \equiv \frac{1}{c^2}\boldsymbol{F}_{\nu}\left(\boldsymbol{r},t\right)$$

monochromatic momentum density *of the radiation field*

$$\left[G_{\nu}\right] = \left(M\,T^{-2}\right) L^{-2}\, T^{2} = \left(M\,L\,T^{-1}\right) L^{-3}\, T$$

$$\boldsymbol{G} \equiv \int_0^\infty \boldsymbol{G}_\nu \, d\nu = \frac{1}{c^2} \int_0^\infty \boldsymbol{F}_\nu \, d\nu = \frac{1}{c^2} \boldsymbol{S}$$

To cut a long story short:

$$\frac{\partial (\boldsymbol{G}_\nu)_j}{\partial t} = \frac{1}{c^2} \frac{\partial (\boldsymbol{F}_\nu)_j}{\partial t} = -\nabla \cdot \left[(\boldsymbol{G}_\nu)_j \, c \, \boldsymbol{n} \right]$$

continuity equation

$$\frac{\partial}{\partial t} \int_0^\infty \boldsymbol{G}_\nu \, d\nu = -\nabla \cdot \int_0^\infty \underline{\underline{\boldsymbol{T}}}_\nu \, d\nu \quad => \quad \frac{\partial \boldsymbol{G}}{\partial t} = -\nabla \cdot \underline{\underline{\boldsymbol{T}}}$$

$$\left(M \, L \, T^{-1} \right) L^{-3} \, T^{-1}$$

momentum density *associated with the* ***electromagnetic field****:*

$$\frac{\partial \boldsymbol{G}_{em}}{\partial t} = \nabla \cdot \underline{\underline{\boldsymbol{T}}}^{M} \qquad \left(M \, L \, T^{-1} \right) L^{-3} \, T^{-1}$$

$$\boldsymbol{G} <=> \boldsymbol{G}_{em}$$

Finis coronat operam