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On spectral radius of signed graphs without negative even cycles

by
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Abstract

We consider the spectral radius of signed graphs without negative even cycles and its relations with the spectral radius of signed graphs obtained by removing a vertex, removing an edge or reversing the sign of an edge. As an application, we determine signed graphs that maximize the spectral radius in the class of unicyclic signed graphs with fixed order and girth. We also give certain upper bounds on the spectral radius of unicyclic oriented graphs.

Key Words: Signed graph, adjacency matrix, largest eigenvalue, spectral radius, unicyclic graph, oriented graph.
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1 Introduction

For a graph $G = (V, E)$, a *signed graph* $\dot{G}$ is a pair (G, σ), where σ is the *signature* satisfying $\sigma(ij) \in \{1, -1\}$, for every $ij \in E$. We say that G is the *underlying graph* of $\dot{G}$ and denote $n = |V|$. The edge set of $\dot{G}$ consists of positive and negative edges, and we interpret a graph as a signed graph with all edges being positive.

The *adjacency matrix* $A_{\dot{G}} = (a_{ij})$ is obtained from the standard $(0, 1)$-adjacency matrix of G by reversing the sign of all 1s which correspond to negative edges. The *eigenvalues* and the *spectrum* of $\dot{G}$ are identified as the eigenvalues and the spectrum of $A_{\dot{G}}$, respectively. Since $A_{\dot{G}}$ is symmetric, its eigenvalues are real. The largest eigenvalue is called the *index* and denoted by λ_1. Similarly, the largest modulus of the eigenvalues of $\dot{G}$ is called the *spectral radius* and denoted by ρ. For a graph G, we have $\lambda_1(G) = \rho(G)$.

In this paper we prove that the spectral radius of a connected signed graph without negative even cycles strictly decreases by either removing a vertex, removing an edge or reversing the sign of an edge belonging to an even cycle. As an application, we determine certain upper bounds for the spectral radius of a unicyclic signed graph and also for the spectral radius of an unicyclic oriented graph. Section 2 contains some preliminaries and known results. Our results are presented in the remaining sections.

2 Preliminaries

For $U \subseteq V(G)$, let $\dot{G}^U$ be the signed graph obtained from $\dot{G}$ by reversing the sign of each edge between a vertex in U and a vertex in $V(\dot{G}) \setminus U$. We say that $\dot{G}$ and $\dot{G}^U$ are *switching* *equivalent*. Switching equivalent signed graphs share the same spectrum. Throughout the paper we do not make any distinction between switching equivalent nor between isomorphic signed graphs.

A signed cycle with l edges is denoted by $\dot{C}_l$. We say that $\dot{C}_l$ is odd (resp. even) if l is odd (even). A cycle in a signed graph is called *positive* if it contains an even number of negative edges, it is said to be *negative*, otherwise. A signed graph is called *balanced* if all its cycles are positive; and *unbalanced*, otherwise. We know from [10] that every switching equivalence class is uniquely determined by the set of positive cycles.

The *bipartite double* $\mathrm{bd}(\dot{G})$ of a signed graph $\dot{G}$ with vertex set $\{i_1, i_2, \ldots, i_n\}$ has the vertex set $\{i_1 1, i_1 2, i_2 1, i_2 2, \ldots, i_n 1, i_n 2\}$ and there is a positive (resp. negative) edge between $i_u j$ and $i_v k$ if and only if there is a positive (negative) edge between i_u and i_v and $j \neq k$. The adjacency matrix of $\mathrm{bd}(\dot{G})$ is determined by the Kronecker product $A_{\mathrm{bd}(\dot{G})} = A_{\dot{G}} \otimes A_{K_2}$, where K_2 is the complete graph with two vertices. The bipartite double $\mathrm{bd}(\dot{G})$ is bipartite and it is connected if and only if $\dot{G}$ is connected and non-bipartite. Observe that the spectral radius of a signed graph is equal to the spectral radius of its bipartite double.

We now proceed by introducing oriented graphs which are considered in Section 5. There is an obvious analogy between them and signed graphs, and many notions are defined analogously. So, for a graph $G = (V, E)$, an *oriented graph* G' is a pair (G, σ'), where σ' is the *orientation* satisfying $\sigma'(ij) \in \{i, j\}$, for every $ij \in E$.

The *adjacency matrix* (or the *skew adjacency matrix*) $S_{G'} = (s_{ij})$ is the $n \times n$ matrix defined by

$$s_{ij} = \begin{cases} 0 & \text{if } ij \notin E, \\ 1 & \text{if } \sigma'(ij) = j, \\ -1 & \text{if } \sigma'(ij) = i. \end{cases}$$

The *eigenvalues* of G' are the eigenvalues of $S_{G'}$ and they form the *spectrum* of G', which consists of purely imaginary numbers which are paired as conjugates. Consequently, the number of non-zero eigenvalues is even. The spectral radius is defined and denoted as in the case of signed graphs.

We say that an even oriented cycle C'_{2l} is *oriented uniformly* if by traversing the cycle we pass through an odd (even) number of edges oriented in the route direction for l odd (even). The bipartite double of an oriented graph is defined and denoted in the same way, so it is determined by $S_{\mathrm{bd}(G')} = S_{G'} \otimes A_{K_2}$. Again, $\rho(G') = \rho(\mathrm{bd}(G'))$.

For an oriented graph $G' = (G, \sigma')$ and a signed graph $\dot{G} = (G, \sigma)$, we say that the signature σ is *associated* with the orientation σ' if

$$\sigma(ik)\sigma(jk) = s_{ik}s_{jk} \text{ holds for every pair of edges } ik \text{ and } jk. \tag{2.1}$$

This association is a symmetric relation.

We know from [8] that, for a graph G and an orientation σ', there exists a signature σ associated with σ' if and only if G is bipartite. The orientation σ' and the signature σ are associated in the sense that σ' induces two switching equivalent signed graphs, one with signature σ and the other with signature $-\sigma$, and the signature σ induces two oriented graphs, one with orientation σ' and the other obtained by reversing the orientation of every edge of the first one.

Clearly, the orientation of an even cycle induces (in the sense of (2.1)) the signature with an even number of negative edges if and only if the cycle is oriented uniformly.

We conclude this section by a recently established relation between the spectrum of an oriented graph and the spectrum of the corresponding signed graph. The exponent stands for the multiplicity of the corresponding eigenvalue.

Theorem 1. [8] *For a bipartite graph G and an orientation σ', if* $\operatorname{rank}(S_{G'}) = 2k$ *and σ is associated with σ', then*

$$\pm i\lambda_1, \pm i\lambda_2, \dots, \pm i\lambda_k, 0^{n-2k}$$

are the eigenvalues of $G' = (G, \sigma')$ if and only if

$$\pm\lambda_1, \pm\lambda_2, \dots, \pm\lambda_k, 0^{n-2k}$$

are the eigenvalues of $\dot{G} = (G, \sigma)$.

Theorem 2. [8] *Given a graph G and an orientation which determines G' such that* $\operatorname{rank}(S_{G'}) = 2k$, *let $H' = (H, \sigma')$ denote the bipartite double of G' and $\dot{H} = (H, \sigma)$ denote the signed graph whose signature is associated with σ'. Then*

$$\pm i\lambda_1, \pm i\lambda_2, \dots, \pm i\lambda_k, 0^{n-2k}$$

are the eigenvalues of G' if and only if

$$(\pm\lambda_1)^2, (\pm\lambda_2)^2, \dots, (\pm\lambda_k)^2, 0^{2(n-2k)}$$

are the eigenvalues of $\dot{H}$.

Both results are proved on the basis of the fact that if σ is associated with σ', then $-S_{G'}^2 = A_{\dot{G}}^2$, whose proof relies on the following chain of equalities

$$-(s_{ij})^2 = -\sum_{k=1}^{n} s_{ik}s_{kj} = \sum_{k=1}^{n} s_{ik}s_{jk} = \sum_{k=1}^{n} \sigma(ik)\sigma(jk) = \sum_{k=1}^{n} a_{ik}a_{kj} = (a_{ij})^2,$$

where the last but one equality follows by the symmetry of $A_{\dot{G}}$. For more details, see the corresponding reference.

3 The spectral radius of signed graphs without negative even cycles

We start with the following lemma.

Lemma 1. *Let $\dot{C}_l$ be a cycle of length l in a signed graph $\dot{G}$.*

(i) If l is odd, then $\dot{C}_l$ produces a positive cycle of length $2l$ in $\operatorname{bd}(\dot{G})$.

(ii) If l is even, then $\dot{C}_l$ produces its two edge-disjoint copies in $\operatorname{bd}(\dot{G})$.

Conversely, let $\dot{C}_{2l}$ be a cycle in $\operatorname{bd}(\dot{G})$.

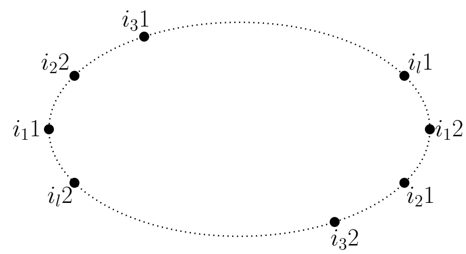

Figure 1: A sketch of the signed cycle $\dot{C}_{2l}$ for the proof of Lemma 1(i).

(i') If l is odd, then either $\dot{C}_{2l}$ is positive and arises from a cycle of length l of $\dot{G}$ or there exists its edge-disjoint copy in $\mathrm{bd}(\dot{G})$ *and both of them arise from the isomorphic cycle of $\dot{G}$.*

(ii') If l is even, then there exists its edge-disjoint copy in $\mathrm{bd}(\dot{G})$ *and both of them arise from the isomorphic cycle of $\dot{G}$.*

Proof. For (i), if the vertices of $\dot{C}_l$ are labelled by $i_1, i_2, \ldots, i_l$ and indexed in the natural order, then the edge $i_u i_v$ of this signed cycle gives its two copies, $i_u 1 i_v 2$ and $i_u 2 i_v 1$, in the corresponding cycle of $\dot{C}_{2l}$ as sketched in Figure 1. In other words, a negative edge of $\dot{C}_l$ gives rise to two negative edges of $\dot{C}_{2l}$, which means that the latter cycle is positive.

Cases (ii) and (ii') follow by definition of a bipartite double, while (i') follows by the same definition and (i) of this lemma. □

We now obtain the relation between the spectral radius of signed graphs without negative even cycles and the spectral radius of their vertex-deleted or edge-deleted subgraphs.

Theorem 3. *Let $\dot{G}$ be a connected signed graph without negative even cycles.*

(i) For $v \in V(\dot{G})$, $\rho(\dot{G} - v) < \rho(\dot{G})$;

(ii) For $e \in E(\dot{G})$, $\rho(\dot{G} - e) < \rho(\dot{G})$.

Proof. For (i), if $\dot{G}$ is bipartite, then since all the cycles are positive, $\dot{G}$ and $\dot{G} - v$ are switching equivalent to their underlying graphs, and therefore $\rho(\dot{G} - v) < \rho(\dot{G})$ is a consequence of the Perron-Frobenius theory for simple graphs. If $\dot{G}$ is non-bipartite, then its spectral radius is equal to the index of its bipartite double. Since, by Lemma 1, all the cycles of $\mathrm{bd}(\dot{G})$ are positive, we get that $\mathrm{bd}(\dot{G})$ is switching equivalent to its underlying graph. Since $\mathrm{bd}(\dot{G} - v)$ is obtained by removing the two copies of v in $\mathrm{bd}(\dot{G})$, the result follows in the same way as before.

The proof of (ii) is a slight modification of that of (i). □

Let $\dot{G} * e$ denote the signed graph obtained by reversing the sign of the edge e of $\dot{G}$.

Theorem 4. *Let $\dot{G}$ be a connected signed graph without negative even cycles. If e is an edge that belongs to at least one even cycle, then $\rho(\dot{G} * e) < \rho(\dot{G})$.*

Proof. If $\dot{G}$ is bipartite, then it is switching equivalent to G, and the result follows immediately.

If $\dot{G}$ is non-bipartite, we recall from [7] that the index of a connected signed graph is less than or equal to the index of its underlying graph, with equality if and only if they are switching equivalent. Now, by the assumption of this theorem and Lemma 1(i)&(ii), $\mathrm{bd}(\dot{G})$ is switching equivalent to its underlying graph, since it does not contain a negative cycle. Similarly, $\mathrm{bd}(\dot{G} * e)$ is not switching equivalent to its underlying graph, since by Lemma 1(ii), every cycle that arises from an even cycle of $\dot{G}$ which contains e is negative. As $\mathrm{bd}(\dot{G})$ and $\mathrm{bd}(\dot{G} * e)$ share the same underlying graph (that is $\mathrm{bd}(G)$), we get

$$\rho(\dot{G} * e) = \lambda_1(\mathrm{bd}(\dot{G} * e)) < \lambda_1(\mathrm{bd}(G)) = \lambda_1(\mathrm{bd}(\dot{G})) = \rho(\dot{G}),$$

which completes the proof. □

4 The spectral radius of unicyclic signed graphs

A connected signed graph is called *unicyclic* if the number of its vertices is equal to the number of its edges. The length of the unique cycle is called a *girth* and denoted by g.

Let C_n^g denote the unicyclic graph obtained by attaching $n-g$ pendant vertices at a fixed vertex of a cycle C_g. We know from [2, 5] that the maximum index in the class of unicyclic graphs with fixed number of vertices and fixed girth is attained for C_n^g, and this extremal graph is unique. By the same references, the maximum index in the class of unicyclic graphs with fixed number of vertices is attained uniquely for C_n^3. Using the Schwenk formula [3, Theorem 2.3.4], we get that $\lambda_1(C_n^g)$ is equal to the largest root of

$$(x^2 - (n-g))\Phi_{P_{g-1}}(x) - 2x(\Phi_{P_{g-2}}(x) - 1),$$

where $\Phi_{P_{g-1}}$ stands for the characteristic polynomial of the path with $g-1$ vertices. We denote this root by $\zeta = \zeta(n, g)$; it can be approximated by means of numerical mathematics. In particular, $\zeta(n, 3)$ is the largest root of $x^4 - nx^2 - 2x + n - 3$, while

$$\zeta(n, 4) = \sqrt{\frac{n + \sqrt{n^2 - 8n + 32}}{2}}. \tag{4.1}$$

Theorem 5. *Let $\dot{G}$ denote a unicyclic signed graph with n vertices and girth g.*

(i) If $\dot{G}$ is balanced, then $\rho(\dot{G}) \leq \zeta(n, g)$, with equality if and only if $\dot{G}$ is switching equivalent to C_n^g.

(ii.a) If $\dot{G}$ is unbalanced and g is even, then $\rho(\dot{G}) < \rho(C_n^g) = \zeta(n, g)$.

(ii.b) If $\dot{G}$ is unbalanced and g is odd, then $\rho(\dot{G}) < \rho(C_n^g) = \zeta(n, g)$ and also $\rho(\dot{G}) \leq \zeta(2n, 2g)$, with equality if and only if $n = g$.

Proof. Since if $\dot{G}$ is balanced, then it is switching equivalent to its underlying graph, we deduce that case (i) follows by the discussion that precedes this theorem (including the mentioned result of [2]).

Case (ii.a) and the first inequality of (ii.b) follow by the same discussion and Theorem 4 since, in both situations, $\rho(\dot{G}) < \rho(G)$.

For the second inequality of (ii.b), by definition of a bipartite double, we have that $\mathrm{bd}(\dot{G})$ is also unicyclic with $2n$ vertices and girth $2g$, while by Lemma 1(i) we have that it is balanced. Thus, $\rho(\dot{G}) = \lambda_1(\mathrm{bd}(\dot{G})) = \lambda_1(\mathrm{bd}(G)) \leq \zeta(2n, 2g)$. Equality holds if and only if $\mathrm{bd}(G) \cong C_{2n}^{2g}$, where $\cong$ stands between isomorphic (signed) graphs, which, again by definition of a bipartite double, holds if and only if G is a cycle, and we are done. ∎

There is a similar recent result obtained in [1] which states that among all unbalanced unicyclic signed graphs the maximum index is attained for the unbalanced triangle with all remaining vertices being attached at the same vertex of the triangle.

We have the following corollary.

Corollary 1. *Let $\dot{G}$ be a unicyclic signed graph with odd (resp. even) girth. Then $\rho(\dot{G}) \leq \zeta(n, 3)$ ($\rho(\dot{G}) \leq \zeta(n, 4)$), with equality if and only if $\dot{G}$ is switching equivalent to C_n^3 (C_n^4).*

Proof. The first inequality follows by the previous theorem. The other one follows since, for g even, we have $\zeta(n, g) \leq \zeta(n, 4)$, with equality if and only if $g = 4$, which is a direct consequence of the result obtained by Hoffman and Smith concerning the spectral radius of a graph with an internal path [4]; the 'signed' version of this result is given in [2]. ∎

5 An upper bound for the spectral radius of a unicyclic oriented graph

Here is an upper bound for $\rho(G')$ expressed in terms of $\zeta(n, g)$.

Theorem 6. *For an oriented unicyclic graph G' with n vertices and girth g,*

$$\rho(G') \begin{cases} < \zeta(n, g) & \text{if } g \text{ is odd} \\ \leq \zeta(n, g) & \text{if } g \text{ is even,} \end{cases}$$

with equality (for g even) if and only if $G \cong C_n^g$ and the cycle of G' is oriented uniformly.

Proof. If g is even, then G' is bipartite, and then by Theorem 1, its spectral radius is equal to the spectral radius of the associated signed graph. The result follows by Theorem 5(i).

If g is odd, then we make a tour from G' to C_n^g realized in the following way and visualized in Figure 2. Let $\dot{H}$ be the (bipartite) signed graph associated with $\mathrm{bd}(G')$, and let H be the underlying graph of $\dot{H}$. It can be easily verified that H is also a bipartite double of G (the underlying graph of G'), and so we have

$$\rho(G') = \rho(\mathrm{bd}(G')) = \rho(\dot{H}) < \rho(H) = \lambda_1(G) \leq \lambda_1(C_n^g) = \zeta(n, g),$$

where all the equalities and the inequalities in the above chain are clear, except possibly the strict inequality, which follows by the following arguments. By definition of a bipartite double, we easily conclude that the unique cycle of $\mathrm{bd}(G')$ is not oriented uniformly as it contains an even number of edges oriented in the route direction, which means that the corresponding cycle of $\dot{H}$ is negative, which implies the strict inequality. ∎

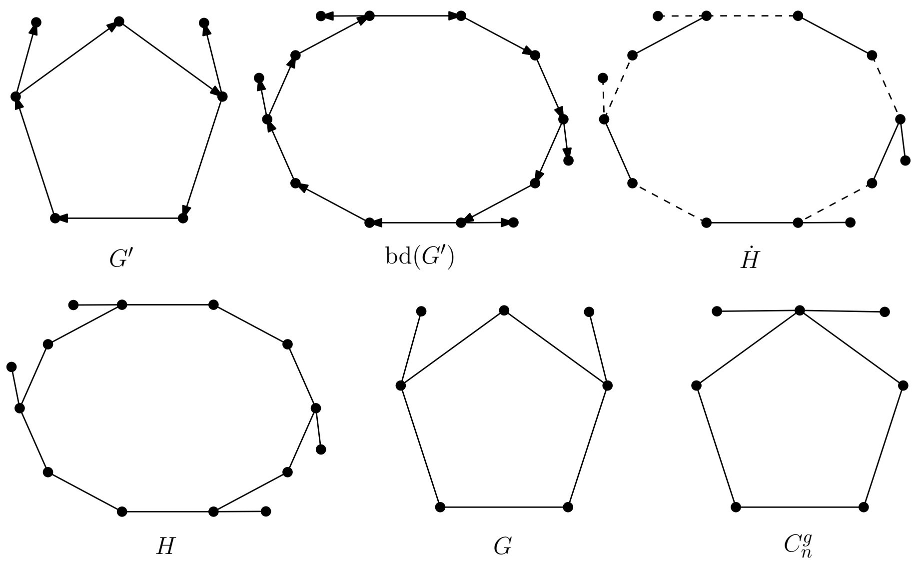

Figure 2: A visualization of a tour from G' to C_n^g via bd(G'), $\dot{H}$, H and G, for the proof of Theorem 6. Negative edges of $\dot{H}$ are dashed.

Here is an immediate consequence.

Corollary 2. *For an oriented unicyclic graph G' with n vertices and girth g,*

$$\rho(G')\begin{cases} <\zeta(n,3) & \text{if } g \text{ is odd} \\ \leq \zeta(n,4) & \text{if } g \text{ is even,} \end{cases} \tag{5.1}$$

with equality (for g even) if and only if $G \cong C_n^4$ and the cycle of G' is oriented uniformly.

Proof. The strict inequality follows since $\zeta(n,g) < \zeta(n,3)$, while the other one follows as in the proof of Corollary 1. □

In [9], Xu obtained a sharp upper bound for the spectral radius of a unicyclic oriented graph G' stating that

$$\rho(G') \leq \sqrt{\frac{n+\sqrt{n^2-4n+12}}{2}} \tag{5.2}$$

with equality if and only if $G \cong C_n^3$. In fact, the right-hand side is identified with the spectral radius of an oriented graph obtained by defining an arbitrary orientation on C_n^3. Since $\zeta(n,3)$ is greater than the right-hand side of (5.2), we deduce that (5.2) gives a better estimate than (5.1), whenever g is odd, but one may observe that $\zeta(n,3)$ is asymptotically the upper bound of (5.2). On the other hand, for g even, by comparing (4.1) and (5.2), we get that the bounds of (5.1) and (5.2) are equal for $(n,g) = (5,4)$, while in any other case the former one is finer.

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