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On Quasiconformal Harmonic Surfaces with Rectifiable Boundary

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Abstract It is proved that every quasiconfomal harmonic mapping of the unit disk onto a surface with rectifiable boundary has absolutely continuous extension to the boundary as well as its inverse mapping has this property. In addition it is proved an isoperimetric type inequality for the class of these surfaces. These results extend some classical results for conformal mappings, minimal surfaces and surfaces with constant mean curvature treated by Kellogg, Courant, Nitsche, Tsuji, F. Riesz and M. Riesz, etc.

Keywords Quasiconformal maps · Harmonic surfaces · Rectifiable boundary

Mathematics Subject Classification (2000) Primary 30C65; Secondary 31B05

1 Introduction and Background

A mapping $u = (u_1, \dots, u_n) : D \to \mathbf{R}^n$ is called *harmonic* in a region $D \subset \mathbf{C}$ if for $k = 1, \dots, n$, u_k is real-valued harmonic functions in D; that is u_k is twice

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differentiable and satisfies the Laplace equation

$$\Delta u_k := u_{kxx} + u_{kyy} = 0.$$

Let

$$P(r, x) = \frac{1 - r^2}{2\pi(1 - 2r\cos x + r^2)}$$

denote the Poisson kernel. Then every bounded harmonic mapping $u : \mathbf{U} \to \mathbf{R}^n$, $n \geq 1$, defined on the unit disc $\mathbf{U} := \{z : |z| < 1\}$ has the following representation

$$u(z) = P[f](z) = \int_0^{2\pi} P(r, x - \varphi) f(e^{ix}) dx, \tag{1.1}$$

where $z = re^{i\varphi}$ and f is a bounded integrable function defined on the unit circle S^1.

The Hardy space H^p (h^p) for $0 < p < \infty$ is the class of holomorphic functions $f : \mathbf{U} \to \mathbf{C}$ (harmonic mappings $u : \mathbf{U} \to \mathbf{R}^n$) on the open unit disk satisfying

$$\sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \, d\theta \right)^{\frac{1}{p}} < \infty.$$

$$\left(\sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left\| u(re^{i\theta}) \right\|^p \, d\theta \right)^{\frac{1}{p}} < \infty \right).$$

By $\| \cdot \|$ is denoted the Euclidean norm in $\mathbf{R}^n$.

A diffeomorphism $\varphi : \Omega \to \Omega'$, where $\Omega, \Omega' \subset \mathbf{C}$ is called K ($K \geq 1$) quasiconformal if

$$\frac{|\varphi_{\bar{z}}|}{|\varphi_z|} \leq \frac{K - 1}{K + 1},$$

or what is the same if

$$|\varphi_z|^2 + |\varphi_{\bar{z}}|^2 \leq \frac{1}{2} \left(K + \frac{1}{K} \right) (|\varphi_z|^2 - |\varphi_{\bar{z}}|^2).$$

Note that, in our context is enough to assume that φ is diffeomorphism, however, the classical definition of quasiconformality assumes weaker conditions (cf. [1, pp. 3, 23–24]). See also [27].

Let X be a conformal mapping of the unit disk onto a smooth two-dimensional surface $M^2 \subset \mathbf{R}^n$. A diffeomorphism $u : \mathbf{U} \to M^2 \subset \mathbf{R}^n$ is called K quasiconformal

($K \geq 1$) if the mapping $\varphi := X^{-1} \circ u$ is K quasiconformal. It can be easily shown that a mapping u is K quasiconformal if and only if

$$\|u_x\|^2 + \|u_y\|^2 \leq \left(K + \frac{1}{K}\right)\left(\|u_x\|^2 \cdot \|u_y\|^2 - \langle u_x, u_y\rangle^2\right)^{1/2}, \quad z = x + iy \in \mathbf{U}, \tag{1.2}$$

where by $\langle \cdot, \cdot \rangle$ is denoted the standard inner product in the space $\mathbb{R}^n$. If $K = 1$ then (1.2) is equivalent to the system of the equations

$$\|u_x\|^2 = \|u_y\|^2 \text{ and } \langle u_x, u_y\rangle = 0, \tag{1.3}$$

which represent isothermal (conformal) coordinates of the surface M^2. If u is harmonic and satisfies the system (1.3) then M^2 is a minimal surface. We will consider harmonic quasiconformal mappings between surfaces and investigate their character at the boundary. In some recent results, see [8–15,17,18,21–23] is established the Lipschitz and bi-Lipschitz character of these mappings, providing that the boundary is smooth enough (for example if it is C^2). In this paper we study differentiable character of quasiconformal harmonic mappings, assuming that the boundary of the surface is less regular, more precisely we assume that it is rectifiable. If w is a conformal mapping of the unit disk onto a domain with rectifiable Jordan boundary, then w as well as its inverse, has an absolutely continuous extension to the boundary and its boundary function maps the null sets onto null sets (a result of Riesz [25]). This result has been extended to the minimal surface by Tsuji [29]. Concerning quasiconformal mapping in the plane Ahlfors and Beurling showed that the boundary function of a quasiconformal mapping of the unit disk onto itself need not be absolutely continuous [2]. The similar answer has been given by Heinonen for quasiconformal mappings in the space [5,6]. For boundary properties of minimal surfaces we refer to papers of Nitsche [19,20]. Since quasiconformal harmonic mappings are generalizations of conformal mappings and quasiconformal harmonic surfaces are generalization of minimal surfaces, it was intrigue to establish this problem for these class of mappings. For the first class (quasiconformal harmonic mappings between plane domains) the answer is positive and this is a result of Mateljevic, Pavlović, Kalaj, for its proof see for example [7]. See also [21] for the reproduction of this result. In this paper we show that, a harmonic quasiconformal surface $u : \mathbf{U} = \operatorname{int} S^1 \to M^2 = \operatorname{int}\gamma \subset \mathbf{R}^n$ is absolutely continuous on the boundary, as well as its inverse function has this property (Theorem 3.1 and Remark 3.8). Using the previous fact we show that every quasiconformal harmonic mapping is a "real part" of a conformal mapping, representing a minimal surface in C^n with rectifiable boundary. This fact yields an isoperimetric inequality for quasiconformal harmonic surfaces (Theorem 3.5). Next we show that, the null sets in S^1 corresponds to the null sets in γ and the null sets in γ corresponds to the null sets in S^1 (Theorem 3.7).

2 Preliminary Results

Lemma 2.1 *If u is K quasiconformal then for almost every $z \in \mathbf{U}$*

$$\max\{\|\nabla u(z)h\| : |h| = 1\} \leq K \min\{\|\nabla u(z)h\| : |h| = 1\}, \tag{2.1}$$

and

$$\left\| \frac{\partial u}{\partial \varphi} \right\|^2 \geq \frac{r^2}{1+K^2}(\|u_x\|^2 + \|u_y\|^2). \tag{2.2}$$

The condition (2.1) is also sufficient for u being K quasiconformal.

Proof Let $h = (\alpha, \beta) \in S^1$. Then

$$\|\nabla u(z)h\|^2 = \|u_x\|^2\alpha^2 + 2\langle u_x, u_y\rangle \alpha\beta + \|u_y\|^2\beta^2.$$

This yield

$$\max\{\|\nabla u(z)h\| : |h| = 1\} = \sqrt{\frac{(\|u_x\|^2 + \|u_y\|^2)(1+\sqrt{1-4\eta^2})}{2}} \tag{2.3}$$

and

$$\min\{\|\nabla u(z)h\| : |h| = 1\} = \sqrt{\frac{(\|u_x\|^2 + \|u_y\|^2)(1-\sqrt{1-4\eta^2})}{2}}, \tag{2.4}$$

where

$$\eta = \frac{(\|u_x\|^2 \cdot \|u_y\|^2 - \langle u_x, u_y\rangle^2)^{1/2}}{\|u_x\|^2 + \|u_y\|^2}.$$

According to (1.2)

$$\eta \geq \frac{K}{1+K^2}. \tag{2.5}$$

On the other hand

$$\frac{\partial u}{\partial \varphi} = r u_y \cos\varphi - r u_x \sin\varphi, \quad z = re^{i\varphi}. \tag{2.6}$$

Therefore

$$\left\| \frac{\partial u}{\partial \varphi} \right\| \geq r \min\{\|\nabla u(z)h\| : |h| = 1\}. \tag{2.7}$$

Dividing (2.3) and (2.4), and using (2.5) we obtain (2.1). By (2.5), (2.4) and (2.7) we obtain (2.2). □

Assume that $u : \mathbf{U} \to \mathbf{R}^n$ is a harmonic mapping defined in the unit disk $\mathbf{U}$. Consider auxiliary family of mappings $\omega_r(z) = u(rz)$. Then ω_r is harmonic and there holds $\omega_r(z) = P[g_r](z)$ where $g_r(e^{i\varphi}) = u(re^{i\varphi})$. Let

$$l_r = \int_0^{2\pi} ||u(re^{i\theta})||d\theta$$

and choose $\rho, r < 1$. Take $h \in R^n : ||h|| = 1$. Then

$$\begin{aligned}\left|\left\langle \int_0^{2\pi} P(\rho, \theta-\varphi)u(re^{i\theta})d\theta, h\right\rangle\right| &= \left|\int_0^{2\pi} P(\rho, \theta-\varphi)\left\langle u(re^{i\theta}), h\right\rangle d\theta\right| \\ &\le \int_0^{2\pi} P(\rho, \theta-\varphi)|\left\langle u(re^{i\theta}), h\right\rangle|d\theta \\ &\le \int_0^{2\pi} P(\rho, \theta-\varphi)||u(re^{i\theta})||d\theta.\end{aligned}$$

Therefore

$$\left\|\int_0^{2\pi} P(\rho, \theta-\varphi)u(re^{i\theta})d\theta\right\| \le \int_0^{2\pi} P(\rho, \theta-\varphi)\|u(re^{i\theta})\|d\theta. \tag{2.8}$$

By using (2.8) we obtain

$$\begin{aligned}\int_0^{2\pi} ||u(r\rho e^{i\varphi})||d\varphi &= \int_0^{2\pi} \|\omega_r(\rho e^{i\varphi})\|d\varphi = \int_0^{2\pi}\left\|\int_0^{2\pi} P(\rho, \theta-\varphi)u(re^{i\theta})d\theta\right\| d\varphi \\ &\le \int_0^{2\pi}\int_0^{2\pi} P(\rho, \theta-\varphi)\|u(re^{i\theta})\|d\theta d\varphi = \int_0^{2\pi} \|u(re^{i\varphi})\|d\varphi.\end{aligned} \tag{2.9}$$

Thus we proved the following lemma of Rado [24].

Lemma 2.2 *The function $r \mapsto l_r$ considered before is increasing.*

Lemma 2.3 *Assume that $u_1, u_2, \ldots, u_m : \mathbf{U} \to \mathbb{R}^n$ are harmonic mappings in the unit disk $\mathbf{U}$ and continuous in $\overline{\mathbf{U}}$. Then the function $f(z) = \sum_{i=1}^m \|u_i\|$ satisfies the* *maximum principle*

$$f(z) \leq \max_{|z|=1} f(z).$$

Proof Similarly as in (2.8) it can be proved that

$$\|u_i(z)\| \leq \int_0^{2\pi} P(r, \theta - \varphi)\|u_i(e^{i\theta})\| d\theta, \quad z = re^{i\varphi},\ i = 1, \ldots m.$$

From the previous inequalities it follows that

$$\sum_{i=1}^{m} \|u_i(z)\| \leq \sum_{i=1}^{m} \int_0^{2\pi} P(r, \theta - \varphi)\|u_i(e^{i\theta})\| d\theta,$$

and therefore

$$f(z) \leq \max_{\theta \in [0, 2\pi]} \sum_{i=1}^{m} \|u_i(e^{i\theta})\|,$$

as desired. □

3 The Main Results

Theorem 3.1 *If $u(z) = P[F](z)$ is a quasiconformal harmonic mapping of the unit disk $\mathbf{U}$ onto a surface $M^2 \subset \mathbf{R}^n$ bounded by a rectifiable Jordan contour γ, then F is an absolutely continuous function.*

We need the following two propositions.

Proposition 3.2 [26] *For an analytic function f in the unit disk $\mathbf{U}$ to be continuous in $\overline{\mathbf{U}}$ and absolutely continuous in S^1 it is necessarily and sufficient that $f' \in H^1$. If $f' \in H^1$, then for a.e. $\theta \in [0, 2\pi)$ we have*

$$\frac{df(e^{i\theta})}{d\theta} = ie^{i\theta} f'(e^{i\theta}),$$

where

$$f'(e^{i\theta}) := \lim_{r \to 1} f'(re^{i\theta})$$

and $\frac{df(e^{i\theta})}{d\theta}$ is the derivative of the function $\theta \to f(e^{i\theta})$.

Proposition 3.3 [4] *For an analytic function f in the unit disk* $\mathbf{U}$ *to have the representation in* $\mathbf{U}$ *be means of Poisson integral*

$$f(re^{i\varphi}) = \int_0^{2\pi} g(e^{i\theta}) P(r, \varphi - \theta) d\theta,$$

where $g \in L^1(S^1)$ it is necessarily and sufficient that $f \in H^1(\mathbf{U})$.

Proof of Theorem 3.1 Consider the function

$$l_r = \int_0^{2\pi} \left\| \frac{\partial u(re^{i\varphi})}{\partial \varphi} \right\| d\varphi, \quad 0 \leq r < 1.$$

Then $r \mapsto l_r$ is increasing and is equal to the length of the smooth curve $u(S(r))$, where $S(r) = rS^1$. On the other hand side the length of the curve $u(S(r))$ is equal to the limit of the following sequence when $n \to \infty$

$$\begin{aligned} s_r^n(z) = &\left\| u(z) - u(ze^{2\pi i/n}) \right\| + \left\| u(ze^{2\pi i/n}) - u(ze^{4\pi i/n}) \right\| \\ &+ \cdots + \left\| u(ze^{2(n-1)\pi i/n}) - u(z) \right\|, \end{aligned}$$

for every $z \in S(r)$. By using Lemma 2.3, because the mapping u is continuous up to the boundary, we obtain

$$\begin{aligned} s_r^n(z) \leq \max_{\varphi \in [0,2\pi]} &\Big[\| u(e^{i\varphi}) - u(e^{i\varphi} e^{2\pi i/n}) \| + \| u(e^{i\varphi} e^{2\pi i/n}) - u(e^{i\varphi} e^{4\pi i/n}) \| \\ &+ \ldots + \| u(e^{i\varphi} e^{2(n-1)\pi i/n}) - u(e^{i\varphi}) \| \Big]. \end{aligned}$$

Letting $n \to \infty$ (because $u(S^1)$ is a rectifiable curve) we infer that $l_r < l(u(S^1)) < \infty$, where $l(u(S^1))$ denotes the length of $l(u(S^1))$.

Next we have

$$u(z) = (\mathrm{Re}(a_1(z)), \mathrm{Re}(a_2(z)), \ldots, \mathrm{Re}(a_n(z))),$$

where a_i are analytic functions.

Therefore

$$u_x = (\mathrm{Re}(a_1'(z)), \mathrm{Re}(a_2'(z)), \ldots, \mathrm{Re}(a_n'(z))),$$

and

$$u_y = -(\mathrm{Im}(a_1'(z)), \mathrm{Im}(a_2'(z)), \ldots, \mathrm{Im}(a_n'(z))).$$

According to (2.2)

$$\left\|\frac{\partial u}{\partial \varphi}\right\|^2 \geq \frac{1}{1+K^2}\sum_{k=1}^{n}|za_k'(z)|^2. \tag{3.1}$$

It yields that

$$|za_i'(z)| \leq \sqrt{1+K^2}\left\|\frac{\partial u}{\partial \varphi}\right\|.$$

Since

$$\int_0^{2\pi}\left\|\frac{\partial u}{\partial \varphi}\right\| d\varphi \leq \mu(\gamma) < \infty$$

we infer that

$$\frac{\partial u}{\partial \varphi} \in h_1(\mathbf{U}).$$

Therefore for $i = 1, \dots, n$ we have

$$za_i'(z) \in H^1(\mathbf{U})$$

and consequently

$$a_i'(z) \in H^1(\mathbf{U}).$$

By using Propositions 3.2 and 3.3 we obtain that for every $i = 1, \dots, n$ there exists an absolutely continuous function g_i such that

$$a_i(z) = P[g_i(e^{i\theta})](z), \quad i = 1, \dots, n.$$

Therefore

$$u = P[F](z),$$

where

$$F(e^{i\theta}) = (\operatorname{Re} g_1(e^{i\theta}), \dots, \operatorname{Re} g_n(e^{i\theta}))$$

is an absolutely continuous function. □

Corollary 3.4 *It follows from the previous considerations, together with the Fatou's lemma the relation*

$$\lim_{r\to 1} l_r = \int_0^{2\pi} \left\| \frac{\partial F}{\partial \varphi} \right\| d\varphi = l(\gamma).$$

We next prove an isoperimetric type inequality for quasiconformal harmonic surfaces.

Theorem 3.5 *If M^2 is a K quasiconformal harmonic surface spanning a rectifiable curve $\gamma \subset \mathbf{R}^n$ then there hold the following isoperimetric type inequality*

$$\mu(M^2) \le \frac{1+K^2}{2}\frac{1}{4\pi}(l(\gamma))^2, \tag{3.2}$$

where μ denotes area and length respectively. This inequality is asymptotically sharp as $K \to 1$.

Remark 3.6 Courant ([3, Theorem 3.7, page 129]) proved the following inequality for a harmonic surface $M^2 = u(\mathbf{U})$ spanning a rectifiable curve γ

$$\mu\,(M^2) \le \frac{1}{4}(l(\gamma))^2.$$

Proof Since every quasiconformal harmonic surface has the representation

$$u(z) = (\mathrm{Re}(a_1(z)), \mathrm{Re}(a_2(z)), \dots, \mathrm{Re}(a_n(z))) : \mathbf{U} \to \mathrm{int}(\gamma) = M^2 \subset \mathbf{R}^n,$$

it defines a minimal surface with rectifiable boundary

$$w(z) = (a_1(z), a_2(z), \dots, a_n(z)) : \mathbf{U} \to \mathrm{int}(\Gamma) = \Sigma^2 \subset \mathbf{C}^n.$$

Namely

$$||w_x||^2 = \sum_{k=1}^n |a_k'(z)|^2 = \sum_{k=1}^n |i a_k'(z)|^2 = ||w_y||^2$$

and

$$\langle w_x, w_y \rangle = \sum_{k=1}^n \mathrm{Re}(i a_k'(z)\overline{a_k'(z)}) = 0.$$

According to (3.1), the minimal surface w is a surface with rectifiable boundary, and thus it has an absolutely continuous extension to the boundary. It is well-known the following isoperimetric inequality for minimal surfaces

$$A(w) \le \frac{1}{4\pi}L^2(w), \tag{3.3}$$

where

$$A(w) = \frac{1}{2}\int_{\mathbf{U}} (\|w_x\|^2 + \|w_y\|^2)dxdy = \mu(\Sigma^2)$$

and

$$L(w) = \int_{S^1} |dw| = l(\Gamma).$$

Since

$$A(u) = \int_{\mathbf{U}} \sqrt{\|u_x\|^2|u_y|^2 - \langle u_x, u_y\rangle^2}dxdy = \mu(M^2)$$

it follows that

$$\begin{aligned}A(u) &\le \frac{1}{2}\int_{\mathbf{U}} (\|u_x\|^2 + \|u_y\|^2)dxdy\\ &= \frac{1}{4}\int_{\mathbf{U}} (\|w_x\|^2 + \|w_y\|^2)dxdy\\ &= \frac{1}{2}A(w).\end{aligned}$$

Therefore

$$A(u) \le \frac{1}{2}A(w). \tag{3.4}$$

Next,

$$L(u) = \int_{S^1} \left\|\frac{\partial u}{\partial\varphi}\right\| d\varphi = l(\gamma).$$

Since

$$\left\|\frac{\partial w}{\partial\varphi}\right\|^2 = \sum_{k=1}^{n} |za_k'(z)|^2,$$

from (3.1) we obtain

$$\frac{1}{\sqrt{1+K^2}}\left\|\frac{\partial w}{\partial\varphi}\right\| \le \left\|\frac{\partial u}{\partial\varphi}\right\|.$$

Hence

$$\frac{1}{\sqrt{1+K^2}}L(w) \leq L(u). \tag{3.5}$$

From (3.3), (3.4) and (3.5) we infer

$$\frac{A(u)}{L^2(u)} \leq \frac{(1+K^2)}{2}\frac{A(w)}{L^2(w)} \leq \frac{1+K^2}{8\pi},$$

i.e.

$$\mu(M^2) \leq \frac{(1+K^2)}{8\pi}(l(\gamma))^2.$$

□

The next theorem is an extension of the corresponding theorem for minimal surfaces [29].

Theorem 3.7 *If $u = P[F]$ is a quasiconformal harmonic mapping of the unit disk* **U** *onto a surface M^2 with rectifiable boundary γ, then for every Lebesgue measured set $E \subset S^1$, $F(E) \subset \gamma$ is measured and*

$$\mu(E) = 0 \Leftrightarrow \mu(F(E)) = 0.$$

Proof Assume that $\mu(E) = 0$. Prove that $\mu(F(E)) = 0$. Since F is an absolutely continuous function it follows that

$$l(\gamma) = \int_0^{2\pi} \left\| \frac{\partial F}{\partial \varphi} \right\| d\varphi,$$

i.e.

$$\left\| \frac{\partial F}{\partial \varphi} \right\| \in L^1.$$

Thus

$$\left\| \frac{\partial F}{\partial \varphi} \right\| \cdot \chi_E \in L^1$$

and

$$\mu(F(E)) = \int_E \left\| \frac{\partial F}{\partial \varphi} \right\| d\varphi = \int_0^{2\pi} \left\| \frac{\partial F}{\partial \varphi} \right\| \cdot \chi_E d\varphi = 0.$$

Next we will prove that a null set on γ corresponds to a null set on S^1. Let G be a null set on γ which corresponds to E on S^1 and G' be a null set which contains G and is G_δ (G_δ set is a countable intersection of open sets), which corresponds to E' on S^1. Then E' contains E and being the homeomorphic image of G_δ is G_δ and hence is measurable. Since $\mu(G') = 0$, we can cover G' by a sequence of open intervals Δs_n such that $\sum_{k=1}^{\infty} |\Delta s_k| < \varepsilon$ where $|\Delta s_n|$ denotes the arc length of Δs_n. Let $\Delta\theta_n$ correspondents to Δs_n on S^1, then

$$|\Delta s_n| = \int\limits_{\Delta\theta_n} \left\| \frac{\partial F}{\partial \varphi}(e^{i\varphi}) \right\| d\varphi,$$

and therefore

$$\varepsilon > \sum_{k=1}^{\infty} |\Delta s_k| = \sum_{k=1}^{\infty} \int\limits_{\Delta\theta_n} \left\| \frac{\partial F}{\partial \varphi}(e^{i\varphi}) \right\| d\varphi \geq \int\limits_{E'} \left\| \frac{\partial F}{\partial \varphi}(e^{i\varphi}) \right\| d\varphi.$$

Since ε is arbitrary, we have

$$\int\limits_{E'} \left\| \frac{\partial F}{\partial \varphi}(e^{i\varphi}) \right\| d\varphi = 0. \tag{3.6}$$

To continue we make use of (3.1). We have

$$\left\| \frac{\partial F}{\partial \varphi}(e^{i\varphi}) \right\|^2 = \lim_{z \to e^{i\varphi}} \left\| \frac{\partial u}{\partial \varphi}(re^{i\varphi}) \right\|^2 \geq \frac{1}{1+K^2} \sum_{i=1}^{n} |a_i'(e^{i\varphi})|^2.$$

By Luzin–Privalov uniqueness theorem

$$\sum_{i=1}^{n} |a_i'(e^{i\varphi})|^2 > 0, \quad \text{for a.e. } \varphi \in [0, 2\pi]$$

if at least one of the analytic functions a_i is nonconstant.

Therefore

$$\left\| \frac{\partial F}{\partial \varphi}(e^{i\varphi}) \right\| > 0, \text{ for a.e. } \varphi \in [0, 2\pi]. \tag{3.7}$$

From (3.7) and (3.6) we infer $\mu(E') = 0$. Since $E \subset E'$, it follows that $\mu(E) = 0$ as desired. □

Remark 3.8 The previous theorem implies that $u^{-1} : M^2 \to \mathbf{U}$ is absolutely continuous on γ; that is for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{k=1}^{m} |\Delta s_k| < \delta \Rightarrow \sum_{k=1}^{m} |\Delta\theta_k| < \varepsilon.$$

In general, the inverse of an absolutely continuous function need not be absolutely continuous (see e.g. [28]).

If $X : \mathbb{U} \to \Sigma^2$ is a conformal mapping of the unit disk onto a surface $\Sigma^2 \subset \mathbb{R}^n$ and $f : \Sigma^2 \to M^2$ is a quasiconformal mapping, then f is harmonic if and only if $f \circ X : \mathbf{U} \to \Sigma^2$ is harmonic. Using the fact that if $\partial\Sigma^2$ is rectifiable and $f \circ X$ is harmonic q.c. then it is absolutely continuous on the boundary as well as its inverse, we obtain.

Corollary 3.9 *Every harmonic quasiconformal mapping between two Euclidean surfaces Σ^2 and M^2 with rectifiable Jordan boundaries has absolutely continuous extension to the boundary as well as its inverse has this property.*

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