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(K, K')-quasiconformal Harmonic Mappings

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Abstract We prove that a harmonic diffeomorphism between two Jordan domains with C^2 boundaries is a (K, K') quasiconformal mapping for some constants $K \geq 1$ and $K' \geq 0$ if and only if it is Lipschitz continuous. In this setting, if the domain is the unit disk and the mapping is normalized by three boundary points condition we give an explicit Lipschitz constant in terms of simple geometric quantities of the Jordan curve which surrounds the codomain and (K, K'). The results in this paper generalize and extend several recently obtained results.

Keywords Harmonic mappings · Quasiconformal mappings · Distance function · Curvature

Mathematics Subject Classifications (2010) Primary 30C55 · Secondary 31C05

1 Introduction

A function w is called *harmonic* in a region D if it has form $w = u + iv$ where u and v are real-valued harmonic functions in D. If D is simply-connected, then there are two analytic functions g and h defined on D such that w has the representation

$$w = g + \overline{h}.$$

If w is a harmonic univalent function, then by Lewy's theorem (see [24]), w has a non-vanishing Jacobian and consequently, according to the inverse mapping

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theorem, w is a diffeomorphism. If k is an analytic function and w is a harmonic function then $w \circ k$ is harmonic. However $k \circ w$, in general is not harmonic.

By $\mathbf{R}$ we denote the set of real numbers. Throughout this paper, we will use notation $z = re^{i\varphi}$, where $r = |z|$ and $\varphi \in \mathbf{R}$ are polar coordinates and by w_φ and w_r we denote partial derivatives of w with respect to φ and r, respectively. Let

$$P(r, x) = \frac{1 - r^2}{2\pi(1 - 2r\cos x + r^2)}$$

denote the Poisson kernel. Note that every bounded harmonic function w defined on the unit disk $\mathbf{U} := \{z : |z| < 1\}$ has the following representation

$$w(z) = P[f](z) = \int_0^{2\pi} P(r, x - \varphi) f(e^{ix}) dx, \tag{1.1}$$

where $z = re^{i\varphi}$ and f is a bounded integrable function defined on the unit circle $\mathbf{T} := \{z : |z| = 1\}$.

Here and in the remainder of this paper it is convenient to use the convention: if f is complex-valued function defined on $\mathbf{T}$ a.e. we consider also f as a periodic function defined on $\mathbf{R}$ by $f(t) = f(e^{it})$ and vise versa if the meaning of it is clear from the context; we also write $f'(t) = \frac{\partial f(e^{it})}{\partial t}$.

If f is of bounded variation on $[0, 2\pi]$, it follows from Eq. 1.1 that w_φ equals the Poisson-Stieltjes integral of f:

$$w_\varphi(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \varphi - t) df(t).$$

Hence, by Fatou's theorem, the radial limits of w_φ exist almost everywhere and $\lim_{r\to 1-} w_\varphi(re^{i\varphi}) = f_0'(\varphi)$ a.e., where f_0 is the absolutely continuous part of f. In particular if f is absolutely continuous on $[0, 2\pi]$, then

$$w_\varphi' = P[f']. \tag{1.2}$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbf{R}^{2\times 2}.$$

We will consider the matrix norm:

$$|A| = \max\{|Az| : z \in \mathbf{R}^2, |z| = 1\}$$

and the matrix function

$$l(A) = \min\{|Az| : |z| = 1\}.$$

Let D and G be subdomains of the complex plane $\mathbf{C}$, and $w = u + iv : D \to G$ be a function that has both partial derivatives at a point $z \in D$. By $\nabla w(z)$ we denote the matrix $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$. For the matrix ∇w we have

$$|\nabla w| = |w_z| + |w_{\bar z}| \tag{1.3}$$

and

$$l(\nabla w) = ||w_z| - |w_{\bar{z}}||, \tag{1.4}$$

where

$$w_z := \frac{1}{2}\left(w_x + \frac{1}{i}w_y\right) \text{ and } w_{\bar{z}} := \frac{1}{2}\left(w_x - \frac{1}{i}w_y\right).$$

We say that a function $u : D \to \mathbf{R}$ is ACL (absolutely continuous on lines) in the region D, if for every closed rectangle $R \subset D$ with sides parallel to the x and y-axes, u is absolutely continuous on a.e. horizontal and a.e. vertical line in R. Such a function has of course, partial derivatives u_x, u_y a.e. in D.

A sense-preserving homeomorphism $w\colon D \to G$, where D and G are subdomains of the complex plane $\mathbf{C}$, is said to be (K, K')-quasiconformal (or shortly (K, K')-q.c. or q.c.) $(K \geq 1, K' \geq 0)$ if $w \in$ACL and

$$|\nabla w|^2 \leq KJ_w + K' \ \ (z = re^{i\varphi}), \tag{1.5}$$

where J_w is the Jacobian of w given by

$$J_w = |w_z|^2 - |w_{\bar{z}}|^2 = |\nabla w| l(\nabla w). \tag{1.6}$$

Mappings which satisfy Eq. 1.5 arise naturally in elliptic equations, where $w = u + iv$, and u and v are partial derivatives of solutions, cf [5, Chapter XII].

Let Ω be a Jordan domain with rectifiable boundary. We will say that a mapping $f : \overline{\mathbf{U}} \to \overline{\Omega}$ is *normalized* if $f(t_i) = \omega_i$, $i = 0, 1, 2$, where $\{t_0t_1, t_1t_2, t_2t_0\}$ and $\{\omega_0\omega_1, \omega_1\omega_2, \omega_2\omega_0\}$ are arcs of $\mathbf{T}$ and of $\gamma = \partial\Omega$ respectively, having the same length $2\pi/3$ and $|\gamma|/3$ respectively.

We will say that a mapping $f : U \to V$ is *Hölder (Lipschitz) continuous*, if there exists a constant L such that

$$|f(z) - f(w)| \leq L|z - w|^{\alpha}, \ \ z, w \in U,$$

where $0 < \alpha < 1$ $(\alpha = 1)$.

1.1 Background

Let γ be a Jordan curve. By the Riemann mapping theorem there exists a Riemann conformal mapping of the unit disk onto a Jordan domain $\Omega = \operatorname{int} \gamma$. By Caratheodory's theorem it has a continuous extension to the boundary. Moreover if $\gamma \in C^{n,\alpha}$, $n \in \mathbb{N}$, $0 \leq \alpha < 1$, then the Riemann conformal mapping has $C^{n,\alpha}$ extension to the boundary (this result is known as Kellogg's theorem), see [40]. Conformal mappings are quasiconformal and harmonic. Hence quasiconformal harmonic (shortly HQC) mappings are natural generalization of conformal mappings. The class of HQC automorphisms of the unit disk has been first considered by Martio in [29]. Hengartner and Schober have shown that, for a given second dilatation ($a = \overline{f_{\bar{z}}}/f_z$, with $||a|| < 1$) there exist a q.c. harmonic mapping f between two Jordan domains with analytic boundary [9, Theorem 4.1].

Recently there has been a number of authors who are working on the topic. The situation in which the image domain is different from the unit disk firstly has been considered by the first author in [12]. There it is observed that if f is harmonic

K-quasiconformal mapping of the upper half-plane onto itself normlized such that $f(\infty) = \infty$, then $\operatorname{Im} f(z) = cy$, where $c > 0$; hence f is bi-Lipschitz. In [12] (see also [14]) also characterization of HQC automorphisms of the upper half-plane by means of integral representation of analytic functions is given.

Using the result of Heinz [8]: If w is a harmonic diffeomorphism of the unit disk onto itself with $w(0) = 0$, then $|w_z|^2 + |w_{\bar{z}}|^2 \geq \frac{1}{\pi^2}$, it can be shown that, every quasiconformal harmonic mapping of the unit disk onto itself is co-Lipschitz. Further, Pavlović [35], by using the Mori's theorem on the theory of quasiconformal mappings, proved the following intrigue result: every quasiconformal selfmapping of the unit disk is Lipschitz continuous. Partyka and Sakan [34] yield explicit Lipschitz and co-Lipschitz constants depending on a constant of quasiconformality. Using the Hilbert transforms of the derivative of boundary function, the first characterizations of HQC automorphisms of the upper half-plane and of the unit disk have been given in [14, 35]; for further result cf. [30]. Among the other things Knežević and the second author in [22] showed that a q.c. harmonic mapping of the unit disk onto itself is a $(1/K, K)$ quasi-isometry with respect to Poincaré and Euclidean distance. See also the paper of Chen and Fang [2] for a generalization of the previous result to convex domains.

Since the composition of a harmonic mapping and of a conformal mapping is itself harmonic, using the case of the unit disk and Kellogg's theorem, these theorems can be generalized to the class of mappings from arbitrary Jordan domain with $C^{1,\alpha}$ boundary onto the unit disk. However the composition of a conformal and a harmonic mapping is not, in general, a harmonic mapping. This means in particular that the results of this kind for arbitrary image domain do not follow from the case of the unit disk or the upper half-plane and Kellogg's theorem.

Using some new methods the results concerning the unit disk and the half-plane have been extended properly in the papers [13–21, 27, 30] and [31]. In particular, in [15] we show how to apply Kellogg's theorem and that simple proof in the case of the upper half-plane has an analogy for C^2 domain; namely, we prove a version of "inner estimate" for quasi-conformal diffeomorphisms, which satisfies a certain estimate concerning their laplacian. As an application of this estimate, we show that quasi-conformal harmonic mappings between smooth domains (with respect to the approximately analytic metric), have bounded partial derivatives; in particular, these mappings are Lipschitz.

For related results about quasiconformal harmonic mappings with respect to the hyperbolic metric we refer to the paper of Wan [38] and of Marković [28].

Very recently, Iwaniec, Kovalev and Onninen in [10] have shown that, the class of quasiconformal harmonic mappings is also interesting concerning the modulus of annuli in complex plane.

In this paper we study Hölder and Lipschitz continuity of the class of (K, K')-q.c. harmonic mappings between smooth domains. This class contains conformal mappings and quasiconformal harmonic mappings.

In [32] C. B. Morrey proved a local Hölder estimate for quasiconformal mappings in the plane. Such a Hölder estimate was a fundamental development in the theory of quasiconformal mappings, and had very important applications to partial differential equations. Nirenberg in [33] made significant simplifications and improvements to Morrey's work (in particular, the restriction that the mappings involved be $1-1$ was removed), and he was consequently able to develop a rather complete theory for second order elliptic equation with 2 independent variables. Simon [37, Theorem 2.2] (see also Finn-Serrin [3]) obtain a Hölder estimate for (K, K') quasiconformal mappings, which is analogous to that obtained by Nirenberg in [33], but which is applicable to quasiconformal mappings between surfaces in Euclidean space.

Global Hölder continuity of $(K, 0)$-quasiconformal mapping between domains satisfying certain boundary conditions has been extensively studied by many authors and the results of this kind can be considered as generalizations of Mori's theorem (see for example the papers of Gehring and Martio [4] and Koskela et al. [23]).

1.2 Statement of the Main Result

The main result of this paper is the following theorem which can be considered as an extension of Kellogg theorem and results of Martio, Pavlović, Partyka, Sakan and the authors.

Theorem 1.1 (The main theorem) *Suppose that*

(a1) *Ω is a Jordan domain with C^2 boundary and*
(a2) *w is (K, K') -q.c. harmonic mapping between the unit disk and Ω.*

Then

(c1) *w has a continuous extension to $\overline{\mathbf{U}}$, whose restriction to $\mathbf{T}$ we denote by f.*
(c2) *Furthermore, w is Lipschitz continuous on $\mathbf{U}$.*
(c3) *If f is normalized, there exists a constant $L = L(K, K', \partial\Omega)$ (which satisfies the inequality (Eq. 4.14) below) such that*

$$|f'(t)| \le L \text{ for almost every } t \in [0, 2\pi], \tag{1.7}$$

and

$$|w(z_1) - w(z_2)| \le (KL + \sqrt{K'})|z_1 - z_2| \text{ for } z_1, z_2 \in \mathbf{U}. \tag{1.8}$$

Remark 1.2 Note that a C^2 curve satisfies b −chord-arc condition for some b and has bounded curvature and that the constant L in the previous theorem depends only on K, K', κ_0 and b, where κ_0 is its maximal curvature.

The Hilbert transformation of a function χ is defined by the formula

$$H[\chi](\varphi) = -\frac{1}{\pi}\int_{0+}^{\pi} \frac{\chi(\varphi + t) - \chi(\varphi - t)}{2\tan(t/2)}\mathrm{d}t$$

for a.e. φ and $\chi \in L^1(\mathbf{T})$. The facts concerning the Hilbert transformation can be found in [42, Chapter VII]. By using Theorem 1.1 we deduce

Corollary 1.3 *Let h be a harmonic orientation preserving diffeomorphism between two plane Jordan domains Ω and D with C^2 boundaries. Let in addition $\phi : \mathbf{U} \to \Omega$ be a conformal transformation and take $w = h \circ \phi = P[f]$. Then the following conditions are equivalent*

(1) *h is a (K, K')-qc mapping.*
(2) *h is Lipschitz w.r.t. to the Euclidean metric.*

(3) *f is absolutely continuous on* $\mathbf{T}$, $f' \in L^\infty(\mathbf{T})$ *and* $H[f'] \in L^\infty(\mathbf{T})$.

Proof (1) $\Rightarrow$ (2): Note that, by Kellogg's theorem, w is Lipschitz if and only if $h = w \circ \phi^{-1}$ is Lipschitz. Then

$$|\nabla w|^2 = |\nabla h|^2 |\phi'|^2$$
$$\leq K(J_h|\phi'|^2) + K'|\phi'|^2 = KJ_w + K'|\phi'|^2.$$

Thus w is (K, K'_1) quasiconformal with $K'_1 = K\|\phi'\|^2$, where $\|\phi'\| = \sup\{|\phi'(z)| : z \in \mathbf{U}\}$ is finite by Kellogg's theorem. By Theorem 1.1, w is Lipschitz. By using again the Kellogg's theorem, we obtain ϕ^{-1} is Lipschitz and therefore $h = w \circ \phi^{-1}$ is Lipschitz. Thus we established (1) $\Rightarrow$ (2).

To prove (2) $\Rightarrow$ (1), observe first that the condition (2) implies that $|\nabla h|$ is bounded by a constant M. Thus h is $(1, M^2)$ quasiconformal.

Show now that (3) $\Rightarrow$ (2). Since f is absolutely continuous on $\mathbf{T}$, $f' \in L^\infty(\mathbf{T})$ by Eq. 1.2,

$$w_\varphi(z) = P[f'](z) \quad z = re^{i\varphi}. \tag{1.9}$$

As rw_r is harmonic conjugate of w_φ, we have

$$rw_r(z) = P[H[f']](z)\,. \tag{1.10}$$

Hence, since f', $H[f'] \in L^\infty(\mathbf{T})$, by Eqs. 1.9 and 1.10, we find that rw_r and w_φ are bounded on $\mathbf{U}$ and (2) follows by Kellogg's theorem.

We leave to the reader to prove (2) $\Rightarrow$ (3). □

The following proposition makes clear difference between (K, K')-q.c. harmonic mappings and K-q.c. ($(K, 0)$-q.c.) harmonic mappings.

Proposition 1.4 [11, 18] *Under conditions of Corollary* 1.3*, the following conditions are equivalent:*

(1) *h is a K-qc mapping.*
(2) *h is bi-Lipschitz w.r. to the Euclidean metric.*
(3) *f is absolutely continuous on* $\mathbf{T}$ *and* f', $1/l(\nabla h)$, $H[f'] \in L^\infty(\mathbf{T})$.

By using Corollary 1.3 and Proposition 1.4, we obtain that the function given in the following example is a (K, K')-quasiconformal harmonic mapping which is not $(K, 0)$ quasiconformal.

Example 1.5 Let $f(e^{ix}) = e^{i(x+\sin x)}$. Then the mapping $w = P[f]$ is a Lipschitz mapping of the unit disk $\mathbf{U}$ onto itself, because $f \in C^\infty(\mathbf{T})$ and therefore w is (K, K')-quasiconformal for some K and K' (Corollary 1.3) but it is not $(K, 0)$-quasiconformal for any K, because f is not bi-Lipschitz.

The proof of Theorem 1.1 is presented in Section 4. Previously we prove Proposition 2.2 which can be considered as a Caratheodory theorem for $(K, K')-$ quasiconformal mappings and by using this proposition we extend Smirnov theorem for the class of (K, K')-quasiconformal harmonic mappings. By using Proposition 2.2, Heinz-Berenstein theorem (Lemma 4.5), and distance function with respect to image domain we first show that, (K, K') quasiconformal harmonic mappings are Lipschitz continuous, providing that the boundaryes are twice differentiable Jordan curves. The method developed in [20], and Lemma 2.4 (which is a Mori's type theorem for the class of (K, K') quasiconformal mappings) has an important role on finding the quantitative Lipschitz constant, depending only on (K, K'), the domain and image domain, for normalized (K, K') quasiconformal harmonic mappings. By using Theorem 1.1, we prove Corollary 1.3, and this in turn implies that a harmonic diffeomorphism w between smooth Jordan domains is Lipschitz, if and only if w is (K, K') quasiconformal.

2 Global and Hölder Continuity of (K, K')-q.c. Mappings

For $a \in \mathbf{C}$ and $r > 0$, put $D(a, r) := \{z : |z - a| < r\}$ and define $\Delta_r = \Delta_r(z_0) = \mathbf{U} \cap D(z_0, r)$. Denote by k_ρ the circular arc whose trace is $\{\zeta \in \mathbf{U} : |\zeta - \zeta_0| = \rho\}$.

Lemma 2.1 (The length-area principle) *Assume that f is a $(K, K')-$ q.c. on Δ_r, $0 < r < 1$, $z_0 \in \mathbf{T}$. Then*

$$F(r) := \int_0^r \frac{l_\rho^2}{\rho} d\rho \leq \pi K A(r) + \frac{\pi}{2} K' r^2 , \tag{2.1}$$

where $l_\rho = |f(k_\rho)|$ denote the length of $f(k_\rho)$ and $A(r)$ is the area of $f(\Delta_r)$.

Proof Let $I_\rho = \{t \in [0, 2\pi] : z_0 + \rho e^{it} \in k_\rho\}$. Let $l_\rho = |f(k_\rho)|$ denote the length of $f(k_\rho)$. Since, by Eq. 1.3, $|\nabla f(z)| = |f_z| + |f_{\bar z}|$, using polar coordinates and the Cauchy-Schwarz inequality, we have for almost every ρ

$$\begin{aligned} l_\rho^2 = |f(k_\rho)|^2 &= \left(\int_{k_\rho} |f_z dz + f_{\bar z} d\bar z| \right)^2 \\ &\leq \left(\int_{I_\rho} |\nabla f\left(z_0 + \rho e^{i\varphi}\right)| \rho d\varphi \right)^2 \\ &\leq \int_{I_\rho} |\nabla f\left(z_0 + \rho e^{i\varphi}\right)|^2 \rho d\varphi \cdot \int_{I_\rho} \rho d\varphi. \end{aligned}$$

Hence, we have

$$\begin{aligned} \int_0^r \frac{l_\rho^2}{\rho} d\rho &\leq \pi \int_0^r \int_{I_\rho} |\nabla f\left(z_0 + \rho e^{i\varphi}\right)|^2 \rho d\varphi d\rho \\ &\leq \pi K \int_0^r \int_{I_\rho} J_f\left(z_0 + \rho e^{i\varphi}\right) \rho d\varphi d\rho + K' |\Delta_r| \\ &= \pi A(r) K + K' |\Delta_r|, \end{aligned} \tag{2.2}$$

where $A(r)$ is the area of $f(\Delta_r)$, and

$$|\Delta_r| = -\frac{1}{2} r\sqrt{4 - r^2} + r^2 \arcsin \frac{\sqrt{4 - r^2}}{2} + \arcsin\left(\frac{1}{2} r\sqrt{4 - r^2}\right), \quad 0 < r < \sqrt{2}$$

is the area of Δ_r satisfying

$$|\Delta_r| \leq \frac{\pi}{2} r^2. \tag{2.3}$$

Eqs. 2.2 and 2.3 implies Eq. 2.1. □

A topological space X is said to be locally connected at a point x if every neighborhood of x contains a connected open neighborhood. It is easy to verify that the set A is locally connected in $z_0 \in A$ if for every sequence $\{z_n\} \subset A$, which converges to z_0 there exists, for big enough n, connected set $L_n \subset A$ which contains z_0 and z_n such that $\operatorname{diam}(L_n) \to 0$. The set is locally connected if it is locally connected at every point.

Proposition 2.2 (Caratheodory theorem for (K, K') mappings) *Let D be a simply connected domain in $\overline{\mathbb{C}}$ whose boundary has at least two boundary points such that $\infty \notin \partial D$. Let $f : \mathbf{U} \to D$ be a continuous mapping of the unit disk $\mathbf{U}$ onto D and (K, K') quasiconformal near the boundary $\mathbf{T}$.*

Then f has a continuous extension to the boundary if and only if ∂D is locally connected.

Proof Suppose first that f is continuous on $\overline{\mathbf{U}}$. In fact, we prove that if f is continuous on $\mathbf{T}$ then $f(\mathbf{T})$ is locally connected. Let $\{z_n\}$ be a sequence in ∂D which converges to z and $\zeta_n \in f^{-1}(z_n)$. Passing to subsequence, we can suppose that $\zeta_n \to \zeta$. Since f is continuous at ζ, it follows that $f(\zeta) = z$. The arc $\gamma_n = f([\zeta_n, \zeta])$ is connected subset of ∂D which contains z_0 and z_n. Since f is continuous at ζ, $\operatorname{diam}(L_n) \to 0$. Hence ∂D is locally connected.

Note that we can assume without loss of generality that $\infty = f(0)$. Namely, if $f(0) = c_0$, we can consider $J \circ f$ instead of f, where $J(z) = \frac{1}{z - c_0}$; $J \circ f$ also a (K_1, K_1') quasiconformal mapping near the boundary $\mathbf{T}$.

Fix $\zeta_0 \in T$. By k_ρ denote the circular arc whose trace is $\{\zeta \in \mathbf{U} : |\zeta - \zeta_0| = \rho\}$ and let $l_\rho = |f(k_\rho)|$.

Let

$$A(r) = \int_{\Delta_r} J_f(z) dx dy\,.$$

Since $f(\Delta_\sigma)$ is bounded domain, then

$$A(\sigma) < +\infty.$$

From this inequality and Eq. 2.1 it follows

$$\int_0^\sigma \frac{l_\rho^2}{\rho} d\rho < \infty\,. \tag{2.4}$$

Hence there is a sequence $\rho_n \to 0$ with $l_{\rho_n} \to 0$. Let A_n be an end point of γ_{ρ_n} and assume that z_k, z_k' tend to A_n along γ_{ρ_n}. Let in addition l_k be the arc of γ_{ρ_n} joining z_k and z_k', $w_k = f(z_k)$, $w_k' = f(z_k')$ and $\Lambda_k = f(l_k)$. Then $|w_k - w_k'| \leq |\Lambda_k|$. Since $l_{\rho_n} < \infty$ it follows that $|\Lambda_k|$ tends to 0. Therefore $\lim_{k\to\infty} w_k = \lim_{k\to\infty} w_k' = a_n$. Thus the curves $\Gamma_n = f \circ k_{\rho_n}$ have end points a_n, $b_n \in \partial D$ and $|a_n - b_n| \to 0$ (because $\lim_{n\to\infty} l_{\rho_n} = 0$).

Passing to a subsequence, we can assume that a_n,b_n tend to $w_0 \in \partial D$. Since ∂D is locally connected, there exist connected subsets $L_n \subset \partial D$ such that $w_0, a_n, b_n \in L_n$ and the diameter diam (L_n) tends to 0.

Now Γ_n separates D into two connected components, one containing $f(0) = \infty$. Let D_n be bounded component of $D \setminus \Gamma_n$. By following the topological argument of Carleson and Gamelin [1, Theorem 2.1, pp. 6–7] we claim that D_n is contained in a bounded component of $\overline{\mathbb{C}} \setminus (\Gamma_n \cup L_n)$.

Indeed, otherwise there is a simple closed Jordan arc from a fixed point z_0 to ∞ in $\overline{\mathbb{C}} \setminus (\Gamma_n \cup L_n)$, followed by another arc from ∞ to z_0 in D crossing Γ_n exactly at one point; thus we obtain a simple closed Jordan curve in $\overline{\mathbb{C}} \setminus L_n$ which separates points a_n and b_n, contradicting the connectedness of L_n.

Therefore diam $(D_n) \leq$ diam $(\Gamma_n \cup L_n)$, and thus diam $(D_n) \to 0$. This implies that f is continuous at point ζ_0. □

Remark 2.3 If we replace the hypothesis that f is (K, K') in Proposition 2.2 with $f \in W^{1,2}(\mathbf{U})$, for some $0 < r < 1$, then f has also continuous extension to $\overline{\mathbf{U}}$. After we wrote a version of this paper, Vuorinen informed us that results of this type related to Proposition 2.2 has been announced in [26].

Let $\gamma \in C^{1,\mu}$, $0 < \mu \leq 1$, be a Jordan curve and let g be the arc length parameterization of γ and let $l = |\gamma|$ be the length of γ. Let d_γ be the distance between $g(s)$ and $g(t)$ along the curve γ, i.e.

$$d_\gamma(g(s), g(t)) = \min\{|s-t|, (l - |s-t|)\}. \tag{2.5}$$

A closed rectifiable Jordan curve γ enjoys a $b-$ chord-arc condition for some constant $b > 1$ if for all $z_1, z_2 \in \gamma$ there holds the inequality

$$d_\gamma(z_1, z_2) \leq b|z_1 - z_2|. \tag{2.6}$$

It is clear that if $\gamma \in C^{1,\alpha}$ then γ enjoys a chord-arc condition for some $b_\gamma > 1$.

The following lemma is a (K, K')-quasiconformal version of [39, Lemma 1]. Moreover, here we give an explicit Hölder constant $L_\gamma(K, K')$.

Lemma 2.4 *Assume that γ enjoys a chord-arc condition for some $b > 1$. Then for every $(K, K')-$ q.c. normalized mapping f between the unit disk $\mathbf{U}$ and the Jordan domain $\Omega = \mathrm{int}\gamma$ there holds*

$$|f(z_1) - f(z_2)| \leq L_\gamma(K, K')|z_1 - z_2|^\alpha$$

for $z_1, z_2 \in \mathbf{T}$, $\alpha = \frac{1}{K(1+2b)^2}$ and

$$L_\gamma(K, K') = 4(1+2b)2^\alpha \sqrt{\max\left\{\frac{2\pi K|\Omega|}{\log 2}, \frac{2\pi K'}{K(1+2b)^2+4}\right\}}. \tag{2.7}$$

Proof For $a \in \mathbf{C}$ and $r > 0$, put $D(a, r) := \{z : |z - a| < r\}$. It is clear that if $z_0 \in \mathbf{T} = \partial\mathbf{U}$, then, because of normalization, $f(\mathbf{T} \cap \overline{D(z_0, 1)})$ has common points with at most two of three arcs $\omega_0\omega_1$, $\omega_1\omega_2$ and $\omega_2\omega_0$. (Here $\omega_0, \omega_1, \omega_2 \in \gamma$ divide γ into three arcs with the same length such that $f(1) = \omega_0$, $f(e^{2\pi i/3}) = \omega_1$, $f(e^{4\pi i/3}) = \omega_2$, and $\mathbf{T} \cap \overline{D(z_0, 1)}$ do not intersect at least one of three arcs defined by 1, $e^{2\pi i/3}$ and $e^{4\pi i/3}$).

Let $l_\rho = |f(k_\rho)|$ denotes the length of $f(k_\rho)$. Let $I_\rho = \{t \in [0, 2\pi] : z_0 + \rho e^{it} \in k_\rho\}$. Let $\gamma_\rho := f(\mathbf{T} \cap D(z_0, \rho))$ and let $|\gamma_\rho|$ be its length. Assume w and w' are the endpoints of γ_ρ, i.e. of $f(k_\rho)$. Then $|\gamma_\rho| = d_\gamma(w, w')$ or $|\gamma_\rho| = |\gamma| - d_\gamma(w, w')$. If the first case holds, then since γ enjoys the $b-$chord-arc condition, it follows $|\gamma_\rho| \le b|w - w'| \le b l_\rho$. Consider now the last case. Let $\gamma'_\rho = \gamma \setminus \gamma_\rho$. Then γ'_ρ contains one of the arcs $w_0 w_1$, $w_1 w_2$, $w_2 w_0$. Thus $|\gamma_\rho| \le 2|\gamma'_\rho|$, and therefore

$$|\gamma_\rho| \le 2b l_\rho.$$

Using the first part of the proof, it follows that the length of boundary arc γ_r of $f(\Delta_r)$ does not exceed $2bl_r$ which, according to the fact that $\partial f(\Delta_r) = \gamma_r \cup f(k_r)$, implies

$$|\partial f(\Delta_r)| \le l_r + 2bl_r. \tag{2.8}$$

Therefore, by the isoperimetric inequality

$$A(r) \le \frac{|\partial f(\Delta_r)|^2}{4\pi} \le \frac{(l_r + 2bl_r)^2}{4\pi} = l_r^2 \frac{(1+2b)^2}{4\pi}.$$

Employing now Eqs. 2.2 and 2.3 we obtain

$$F(r) := \int_0^r \frac{l_\rho^2}{\rho} d\rho \le K l_r^2 \frac{(1+2b)^2}{4} + \frac{\pi K'}{2} r^2.$$

Observe that for $0 < r \le 1$ there holds $rF'(r) = l_r^2$. Thus

$$F(r) \le KrF'(r) \frac{(1+2b)^2}{4} + \frac{\pi K'}{2} r^2.$$

Let G be the solution of the equation

$$F(r) = KrF'(r) \frac{(1+2b)^2}{4} + \frac{\pi K'}{2} r^2$$

defined by

$$G(r) = \frac{\frac{\pi K'}{2}}{K \frac{(1+2b)^2}{4} + 1} r^2 = \frac{2\pi K'}{K(1+2b)^2 + 4} r^2.$$

It follows that for

$$\alpha = \frac{2}{K(1+2b)^2}$$

there holds

$$\frac{d}{dr} \log([F(r) - G(r)] \cdot r^{-2\alpha}) \ge 0,$$

i.e. the function $[F(r) - G(r)] \cdot r^{-2\alpha}$ is increasing. This yields

$$[F(r) - G(r)] \le [F(1) - G(1)] r^{2\alpha} \le [\pi K |\Omega| - G(1)] r^{2\alpha}.$$

Now for every $r \leq 1$ there exists an $r_1 \in [r/\sqrt{2}, r]$ such that

$$F(r) = \int_0^r \frac{l_\rho^2}{\rho} d\rho \geq \int_{r/\sqrt{2}}^r \frac{l_\rho^2}{\rho} d\rho = l_{r_1}^2 \log \sqrt{2}.$$

Hence

$$l_{r_1}^2 \leq \frac{2\pi K|\Omega| + G(1)\left(r^{2-2\alpha} - 1\right)}{\log 2} r^{2\alpha}.$$

If z is a point with $|z| \leq 1$ and $|z - z_0| = r/\sqrt{2}$, then by Eq. 2.8

$$|f(z) - f(z_0)| \leq (1 + 2b)l_{r_1}.$$

Therefore

$$|f(z) - f(z_0)| \leq H|z - z_0|^\alpha,$$

where

$$H = (1 + 2b)2^{\alpha/2} \sqrt{\max\left\{\frac{2\pi K|\Omega|}{\log 2}, \frac{2\pi K'}{K(1 + 2b)^2 + 4}\right\}}.$$

Thus we have for $z_1, z_2 \in \mathbf{T}$ the inequality

$$|f(z_1) - f(z_2)| \leq 4H|z_1 - z_2|^\alpha. \tag{2.9}$$

□

Remark 2.5 By applying Lemma 2.4, and by using the Möbius transforms, it follows that, if f is an arbitrary $(K, K')-$q.c. mapping between the unit disk $\mathbf{U}$ and Ω, where Ω satisfies the conditions of Lemma 2.4, then $|f(z_1) - f(z_2)| \leq C(f, \gamma, K, K')|z_1 - z_2|^\alpha$ on $\mathbf{T}$.

2.1 A Question

Lemma 2.4 states that, every (K, K') quasiconformal mapping of the unit disk onto a Jordan domain with rectifiable boundary satisfying chord-arc condition is Hölder on the boundary. This can be extended a little bit, for example the lemma remains true if we put $z_1 \in \mathbf{T}$ and $z_2 \in \mathbf{U}$ instead of $z_1, z_2 \in \mathbf{T}$. On the other hand the results of Nirenberg, Finn, Serrin and Simon state that f is Holder continuous in every compact set of the unit disk. It remains an interesting and important open question, does every (K, K') quasiconformal mapping f between the unit disk and a Jordan domain with smooth boundary enjoy Hölder continuity.

3 Smirnov Theorem for (K, K') q.c. Harmonic Mappings

In this section we extend the Smirnov theorem on the theory of conformal mappings to the class of (K, K') quasiconformal harmonic mappings. Let $h^1 = h^1(\mathbf{U})$ and $H^1 = H^1(\mathbf{U})$ be Hardy spaces of harmonic respectively analytic functions defined on the unit disk.

Proposition 3.1 *Let w be a (K, K') quasiconformal harmonic mapping of the unit disk* $\mathbf{U}$ *onto a Jordan domain D. Then $\nabla w \in h^1$ if and only if ∂D is a rectifiable Jordan curve. Moreover, $\nabla w \in h^1$ implies that w is absolutely continuous on* $\mathbf{T}$.

Proof Assume that $\gamma = \partial D$ is a rectifiable Jordan curve. Consider the function

$$l_r = \int_0^{2\pi} \left| \frac{\partial w(re^{i\varphi})}{\partial \varphi} \right| d\varphi, \ 0 \le r < 1.$$

Then, according to Rado's lemma [36] $r \mapsto l_r$ is increasing and is equal to the length of the smooth curve $w(S(r))$, where $S(r) = rS^1$. On the other hand the length of the curve $w(S(r))$ is equal to the limit of the following sequence when $n \to \infty$

$$s_r^n(z) = \left|w(z) - w\left(ze^{2\pi i/n}\right)\right| + \left|w\left(ze^{2\pi i/n}\right) - w\left(ze^{4\pi i/n}\right)\right| + \\ \cdots + \left|w\left(ze^{2(n-1)\pi i/n}\right) - w(z)\right|,$$

for every $z \in S(r)$. From Proposition 2.2 the mapping w is continuous up to the boundary. Since the sum of subharmonic functions is a subharmonic function and the mapping w is continuous up to the boundary, it follows from the maximum principle of subharmonic functions that

$$s_r^n(z) \le \max_{\varphi \in [0,2\pi]} \left[\left|w\left(e^{i\varphi}\right) - w\left(e^{i\varphi}e^{2\pi i/n}\right)\right| + \left|w\left(e^{i\varphi}e^{2\pi i/n}\right) - w\left(e^{i\varphi}e^{4\pi i/n}\right)\right| + \dots \\ + \left|w\left(e^{i\varphi}e^{2(n-1)\pi i/n}\right) - w\left(e^{i\varphi}\right)\right|\right].$$

Letting $n \to \infty$ (because $w(S^1)$ is a rectifiable curve) we infer that $l_r < l(w(S^1)) < \infty$, where $l(w(S^1))$ denotes the length of $l(w(S^1))$.
Next we have

$$w(z) = g(z) + \overline{h(z)}$$

where g and h are analytic functions. From Eq. 1.6 we obtain

$$|\nabla w|^2 \le K|\nabla w| l(\nabla w) + K'.$$

This implies that

$$|\nabla w| \le \frac{Kl(\nabla w) + \sqrt{K^2 l(\nabla w)^2 + 4K'}}{2}$$

and consequently

$$|\nabla w| \le Kl(\nabla w) + \sqrt{K'}. \tag{3.1}$$

For $z = re^{i\varphi}$ we have

$$\frac{\partial w}{\partial \varphi} = rw_y \cos\varphi - rw_x \sin\varphi. \tag{3.2}$$

Thus

$$rl(\nabla w) \le \left|\frac{\partial w}{\partial \varphi}\right| \le r|\nabla w|. \tag{3.3}$$

From Eqs. 1.5, 1.6, 3.1 and 3.3, we deduce that

$$\frac{1}{r^2}\left|\frac{\partial w}{\partial \varphi}\right|^2 \leq KJ_w + K' \quad \left(z = re^{i\varphi}\right), \tag{3.4}$$

and

$$|\nabla w| \leq \frac{K}{r}\left|\frac{\partial w}{\partial \varphi}\right| + \sqrt{K'}. \tag{3.5}$$

According to Eq. 3.5

$$|g'| + |h'| = |\nabla w| \leq \frac{K}{r}\left|\frac{\partial w}{\partial \varphi}\right| + \sqrt{K'}. \tag{3.6}$$

Since

$$\int_0^{2\pi}\left|\frac{\partial w\left(re^{i\varphi}\right)}{\partial \varphi}\right| d\varphi \leq l\left(\gamma\right) < \infty$$

we infer that

$$\frac{\partial w}{\partial \varphi} \in h^1(\mathbf{U}).$$

Therefore we have

$$zg'(z), zh'(z) \in H^1(\mathbf{U})$$

and consequently

$$g'(z), h'(z) \in H^1(\mathbf{U}).$$

Now, it is known from Hardy space theory, that there exist absolutely continuous functions $\tilde{g}$ and $\tilde{h}$ on $\mathbf{T}$, such that

$$g(z) = P[\tilde{g}(e^{i\theta})](z)$$

and

$$h(z) = P[\tilde{h}(e^{i\theta})](z).$$

Therefore

$$w = P[f](z),$$

where

$$f(e^{i\theta}) = \tilde{g}(e^{i\theta}) + \overline{\tilde{h}(e^{i\theta})}$$

is an absolutely continuous function.

To show the converse observe first that the hypothesis $\nabla w \in h^1(\mathbf{U})$ implies that

$$g'(z), h'(z) \in H^1(\mathbf{U}).$$

Next, it is known from Hardy space theory, that w has continuous extension on $\overline{\mathbf{U}}$. Denote by f the restriction of this extension on $\mathbf{T}$. Then f is absolutely continuous and therefore $f' \in L^1(0, 2\pi)$, where $f'(t) = df(e^{it})/dt$. Hence as the above,

$$\frac{\partial w}{\partial \varphi}(z) = P[f'](z)\,.$$

Since F is injective and absolutely continuous we find (see e.g. [6, Chapter X]) that

$$|\gamma| = \int_0^{2\pi} |f'(\varphi)|d\varphi < \infty$$

and therefore γ is a rectifiable Jordan curve. □

4 Lipschitz Continuity of (K, K')-q.c. Harmonic Mappings

In this section we prove Theorem 1.1 which is the main result of the paper. The proof is based on a result of Heinz and Berenstein (Lemma 4.5) and on the estimate Eq. 4.7 which follows from auxiliary results (Lemmas 4.1–4.4):

Lemma 4.1 [19] *Let Ω be a Jordan C^2 domain, $f : \mathbf{T} \to \partial\Omega$ injective continuous parameterization of $\partial\Omega$ and $w = P[f]$. Suppose that $w = P[f]$ is a Lipschitz continuous harmonic function between the unit disk $\mathbf{U}$ and Ω. Then for almost every $e^{i\varphi} \in \mathbf{T}$ we have*

$$\limsup_{r\to 1-0} J_w(re^{i\varphi}) \le \frac{\kappa_0}{2}|f'(\varphi)| \int_{-\pi}^{\pi} \frac{d_\gamma(f(e^{i(\varphi+x)}),\, f(e^{i\varphi}))^2}{x^2}dx, \tag{4.1}$$

where $J_w(z)$ denotes the Jacobian of w at z, $f'(\varphi) := \frac{d}{d\varphi}f(e^{i\varphi})$ and

$$\kappa_0 = \sup_s |\kappa_s|, \tag{4.2}$$

and κ_s is the curvature of γ at the point $g(s)$.

Let d be the distance function with respect to the boundary of the domain Ω: $d(w) = \operatorname{dist}(w, \partial\Omega)$. Let $\Gamma_\mu := \{z \in \Omega : d(z) \le \mu\}$. For basic properties of distance function we refer to [5]. For example $\nabla d(w)$ is a unit vector for $w \in \Gamma_\mu$, and $d \in C^2(\overline{\Gamma_\mu})$, provided that $\partial\Omega \in C^2$ and $\mu \le 1/\sup\{|\kappa_z| : z \in \partial\Omega\}$. We now have.

Lemma 4.2 *Let Ω be a C^2 Jordan domain, $w : \Omega_1 \mapsto \Omega$ be a C^1, (K, K') q.c., $\chi = -d(w(z))$ and $\mu > 0$ such that $1/\mu > \kappa_0 = \operatorname{ess\,sup}\{|\kappa_z| : z \in \partial\Omega\}$.*
Then:

$$|\nabla\chi| \le |\nabla w| \le K|\nabla\chi| + \sqrt{K'} \tag{4.3}$$

in $w^{-1}(\Gamma_\mu)$.

Proof Observe first that ∇d is a unit vector. From $\nabla\chi = -\nabla d \cdot \nabla w$ it follows that

$$|\nabla\chi| \le |\nabla d||\nabla w| = |\nabla w|.$$

For a non-singular matrix A we have

$$\begin{aligned}\inf_{|x|=1} |Ax|^2 &= \inf_{|x|=1} \langle Ax, Ax\rangle = \inf_{|x|=1} \left\langle A^T Ax, x\right\rangle \\ &= \inf\{\lambda : \exists x \neq 0,\ A^T Ax = \lambda x\} \\ &= \inf\{\lambda : \exists x \neq 0,\ AA^T Ax = \lambda Ax\} \\ &= \inf\{\lambda : \exists y \neq 0,\ AA^T y = \lambda y\} = \inf_{|x|=1} |A^T x|^2.\end{aligned} \tag{4.4}$$

Since w is (K, K')-q.c., it follows that

$$|\nabla w|^2 \leq K|\nabla w| l(\nabla w) + K'.$$

This implies that

$$|\nabla w| \leq Kl(\nabla w) + \sqrt{K'}\,.$$

Next we have that $(\nabla\chi)^T = -(\nabla w)^T \cdot (\nabla d)^T$ and therefore for $x \in w^{-1}(\Gamma_\mu)$, we obtain

$$|\nabla\chi| \geq \inf_{|e|=1} |(\nabla w)^T e| = \inf_{|e|=1} |\nabla w\, e| = l(w) \geq \frac{|\nabla w|}{K} - \frac{\sqrt{K'}}{K}.$$

The proof of Eq. 4.3 is completed. □

Lemma 4.3 [18] *Let* $\{e_1, e_2\}$ *be the natural basis in the space* $\mathbf{R}^2$ *and* Ω, Ω_1 *be two* C^2 *domains. Let* $w : \Omega_1 \mapsto \Omega$ *be a harmonic mapping and let* $\chi = -d(w(z))$ *and* $\mu > 0$ *such that* $1/\mu > \kappa_0 = \operatorname{ess\,sup}\{|\kappa_z| : z \in \partial\Omega\}$. *Then*

$$\Delta\chi(z_0) = \frac{\kappa_{\omega_0}}{1 - \kappa_{\omega_0} d(w(z_0))} \left|(O_{z_0} \nabla w(z_0))^T e_1\right|^2, \tag{4.5}$$

where $e_1 \in T_{z_0}$ *and* T_{z_0} *denotes the tangent space at* z_0, $z_0 \in w^{-1}(\Gamma_\mu)$, $\omega_0 \in \partial\Omega$ *with* $|w(z_0) - \omega_0| = \operatorname{dist}(w(z_0), \partial\Omega)$, *and* O_{z_0} *is an orthogonal transformation.*

Since an orthogonal matrix acts as an isometry of Euclidean space, we have $|(O_{z_0} \nabla w(z_0))^T e_1| \leq |\nabla w(z_0)|$. Hence

$$|\Delta\chi(z_0)| \leq \frac{\kappa_{w_0}}{1 - \kappa_{w_0} d(w(z_0)} |\nabla w(z_0)|^2\,. \tag{4.6}$$

Thus, for a fixed number μ such that $1/\mu > \kappa_0$, we have for z near $\partial\Omega_1$ the following estimate:

Lemma 4.4 *Under the above notation, there is* $c > 0$, *such that*

$$|\Delta\chi(z)| \leq c|\nabla w(z)|^2 \textit{ for } z \in w^{-1}(\Gamma_\mu)\,.$$

Lemma 4.5 (Heinz-Berenstein) [7] *Let* $\chi : \overline{\mathbb{U}} \mapsto \mathbb{R}$ *be a continuous function between the unit disc* $\overline{\mathbb{U}}$ *and the real line satisfying the conditions:*

(1) χ *is* C^2 *on* $\mathbb{U}$,
(2) $\chi(\theta) = \chi(e^{i\theta})$ *is* C^2 *and*

(3) $|\Delta\chi| \le a|\nabla\chi|^2 + b$ *on* $\mathbb{U}$ *for some constant* c_0 *(natural growth condition).*

Then the function $|\nabla\chi| = |\text{grad}\,\chi|$ *is bounded on* $\mathbb{U}$.

4.1 Proof of Theorem 1.1

Note first that the statement $(c1)$ of theorem is a special case of Proposition 2.2. Let us now prove $(c2)$: w is Lipschitz continuous. Suppose that μ is a fixed number such that $1/\mu > \kappa_0$ and note that $w^{-1}(\Gamma_\mu) \subset \mathbf{U}$. From Lemmas 4.4 and 4.2, it follows that there exist constants a_1 and b_1 such that

$$|\Delta\chi| \le a_1|\nabla\chi|^2 + b_1 \text{ for } z \in w^{-1}(\Gamma_\mu). \tag{4.7}$$

On the other hand, by Proposition 2.2, w has a continuous extension to the boundary. Therefore for every $t \in \mathbf{T}$, $\lim_{s\to t}\chi(s) = \chi(t) = 0$. Let $\tilde\chi$ be an C^2 extension of the function $\chi|_{w^{-1}(\Gamma_\mu)}$ in $\mathbf{U}$ (by Whitney theorem it exists [41]). Let $b_0 = \max\{|\Delta\tilde\chi(z)| : z \in \mathbf{U} \setminus w^{-1}(\Gamma_{\mu/2})\}$. Then

$$|\Delta\tilde\chi| \le a_1|\nabla\tilde\chi|^2 + b_1 + b_0.$$

Thus the conditions of Lemma 4.5 are satisfied. We conclude that $\nabla\tilde\chi$ is bounded. According to Eq. 4.3, ∇w is bounded in $w^{-1}(\Gamma_\mu)$ and hence in $\mathbf{U}$ as well. Hence, it follows from Lemma 4.2 that w is Lipschitz continuous.

Proof of $(c3)$. Since $w = P[f]$ is Lipschitz, it follows that f is Lipschitz and

$$\operatorname*{ess\,sup}_{0\le\varphi\le 2\pi} |f'(\varphi)| < \infty.$$

In particular f is absolutely continuous and therefore if we use notation $z = re^{i\varphi}$, we find

$$\frac{\partial w}{\partial\varphi}(z) = P[f'](z)\,, \quad z \in \mathbf{U}. \tag{4.8}$$

According to Eq. 4.8 and Lemma 4.1, there exists a set $E \subset [0, 2\pi]$ with zero measure such that $w_\varphi(re^{i\varphi}) \to f'(\varphi)$ as $r \to 1$ and inequality (Eq. 4.1) holds for $\varphi \in [0, 2\pi] \setminus E$. Therefore, for $\varepsilon > 0$ there exists a $t \in [0, 2\pi] \setminus E$ such that

$$\operatorname*{ess\,sup}_{0\le\varphi\le 2\pi} |f'(\varphi)| =: L \le |f'(t)| + \varepsilon. \tag{4.9}$$

Since $|w_\varphi(re^{it})| \to |f'(t)|$ as $r \to 1$, by Eqs. 3.4 and 4.1, we obtain

$$|f'(t)|^2 \le \frac{\kappa_0}{2}K|f'(t)|\int_{-\pi}^{\pi}\frac{d_\gamma(f(e^{i(t+x)}),\ f(e^{it}))^2}{x^2}dx + K'.$$

Now, we use an elementary result: if $a, b \ge 0$, $y^2 \le ay + b$, then $y \le a + \sqrt{b}$. Hence if $C_2 = K\kappa_0/2$, then for β satisfying $0 < \beta < 1$, we have

$$\begin{aligned} L - \varepsilon &\le C_2\int_{-\pi}^{\pi}\frac{d_\gamma(f(e^{i(t+x)}),\ f(e^{it}))^2}{x^2}dx + \sqrt{K'} \\ &\le C_2\int_{-\pi}^{\pi}\frac{d_\gamma(f(e^{i(t+x)}),\ f(e^{it}))^{2-\beta}}{|x|^{2-\beta}}(b_\gamma L)^\beta dx + \sqrt{K'}. \end{aligned} \tag{4.10}$$

Thus

$$(L-\varepsilon)/L^{\beta} \le b_{\gamma} C_2 \int_{-\pi}^{\pi} \frac{d_{\gamma}(f(e^{i(t+x)}), f(e^{it}))^{2-\beta}}{|x|^{2-\beta}} dx + \sqrt{K'}. \tag{4.11}$$

For $\alpha = \frac{1}{K(1+2b_{\gamma})^2}$, choose β, $0 < \beta < 1$, sufficiently close to 1, so that $\sigma = (\alpha - 1)(2-\beta) > -1$. For example, we can choose

$$\beta = 1 - \frac{\alpha}{2-\alpha},$$

and consequently,

$$\sigma = \frac{\alpha}{2-\alpha} - 1.$$

Since f is a normalized mapping, from Lemma 2.4 and Eq. 2.6, we find

$$d_{\gamma}(f(e^{i(t+x)}), f(e^{it})) \le b_{\gamma}|f(e^{i(t+x)}) - f(e^{it})| \le b_{\gamma} L_{\gamma}(K, K')|x|^{\alpha}.$$

Putting this in Eq. 4.11 and then letting $\varepsilon \to 0$, we get

$$L^{1-\beta} \le C_2 \cdot b_{\gamma}(L_{\gamma}(K, K'))^{2-\beta} \int_{-\pi}^{\pi} |x|^{\sigma} dx + \sqrt{K'} = C_3,$$

and hence

$$L \le C_3^{1/(1-\beta)} = C_3^{\frac{2-\alpha}{\alpha}}. \tag{4.12}$$

Further, we use that $w = g + \overline{h}$, where g and h are two analytic functions in $\mathbf{U}$. Since w is Lipschitz continuous, we see that $g' \in H^{\infty}$ and $h' \in H^{\infty}$, where H^{∞} denotes the Hardy space of bounded analytic functions on $\mathbf{U}$. Hence for a.e. $z = e^{i\varphi} \in \mathbf{T}$

$$\begin{aligned}(|h'(z)| + |g'(z)|)^2 &\le K(|g'(z)| - |h'(z)|)(|g'(z)| + |h'(z)|) + K' \\ &\le K|f'(\varphi)|(|g'(z)| + |h'(z)|) + K'.\end{aligned}$$

Let $\Lambda = |h'(z)| + |g'(z)|$. Then

$$\Lambda^2 \le \Lambda L K + K'.$$

Thus for every $z \in \mathbf{U}$

$$|\nabla w(z)| \le \operatorname{ess\,sup}_{|z|=1}\{|g'(z)| + |h'(z)|\} \le KL + \sqrt{K'}. \tag{4.13}$$

This implies Eq. 1.8.

Remark 4.6

a) The previous proof yields the following estimate of a Lipschitz constant L for a normalized $(K, K')-$quasiconformal harmonic mapping between the unit disk and a Jordan domain Ω bounded by a Jordan curve $\gamma \in C^2$ satisfying a $b-$chord-arc condition.

$$L \le \left(K\lambda\kappa_0 b\,(L_{\gamma}(K, K'))^{1+1/\lambda}\pi^{1/\lambda} + \sqrt{K'}\right)^{\lambda}, \tag{4.14}$$

where

$$\alpha = \frac{1}{K(1+2b)^2}, \ \lambda = \frac{2-\alpha}{\alpha},$$

κ_0 is defined by Eq. 4.2 and $L_\gamma(K, K')$ in Eq. 2.7. Thus L depends only on K, K', κ_0 and $b-$chord-arc condition.

See [22, 34, 35] and [13] for estimates, in the special case where γ is the unit circle, and w is $K-$q.c. ($K' = 0$).

b) Notice that, the previous proof did not depend on Kellogg's and Warschawski theorem (that implies that a conformal mapping of the unit disk onto a Jordan domain Ω with $C^{1,\alpha}$ boundary is bi-Lipschitz) nor on Lindelöf theorem in the theory of conformal mappings (see [6] for this topic). For a generalization of Kellogg's theorem we refer to the paper of Lesley and Warschawski [25], where they gave an example of C^1 Jordan domain D, such that the Riemann conformal mapping of the unit disk $\mathbf{U}$ onto D is not Lipschitz. We expect that, the conclusion of Theorem 1.1 remains true, assuming only that the boundary of Ω is $C^{1,\alpha}$. This problem has been overcome for the class of $(K, 0)$-q.c. mappings in [20] by composing by conformal mappings and by using "approximation argument". However, the composition of a (K, K') q.c. mapping and a conformal mapping is not necessarily a (K_1, K_1') q.c. mapping, and it causes further difficulties because the method used in [20] does not work for (K, K') q.c. mappings in general.

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