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On Certain Nonlinear Elliptic PDE and Quasiconformal Maps Between Euclidean Surfaces

David Kalaj · Miodrag Mateljević

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Abstract We mainly investigate some properties of quasiconformal mappings between smooth 2-dimensional surfaces with boundary in the Euclidean space, satisfying certain partial differential equations (inequalities) concerning Laplacian, and in particular satisfying Laplace equation and show that these mappings are Lipschitz. Conformal parametrization of such surfaces and the method developed in our paper (Kalaj and Mateljević, J Anal Math 100:117–132, 2006) have important role in this paper.

Keywords Isothermal coordinates · Harmonic maps · Quasiconformal mappings · PDE · Lipschitz continuous

Mathematics Subject Classifications (2010) Primary 30C65; Secondary 53C43 · 30C35

1 Introduction

By $\mathbb{U}$ we denote the unit disk, by Ω a domain in $\mathbb{R}^2$ and by S a smooth 2-dimensional surface in $\mathbb{R}^l$, $l \geq 3$.

Let f be a smooth mapping between a Jordan domain Ω and a surface S of the Euclidean space $\mathbb{R}^l$. Consider the functional

$$E[f] = \int\int_\Omega |f_x|^2 + |f_y|^2 dxdy \tag{1.1}$$

D. Kalaj (✉)
Faculty of Natural Sciences and Mathematics, University of Montenegro, Cetinjski put b.b., 81000 Podgorica, Montenegro
e-mail: davidk@t-com.me

M. Mateljević
Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Belgrade, Serbia
e-mail: miodrag@matf.bg.ac.rs

The stationary points of the energy integral $E[f]$ satisfy the Euler–Lagrange equation i.e. Laplace equation

$$\Delta f = f_{xx} + f_{yy} = 4 f_{z\bar{z}} \equiv 0. \tag{1.2}$$

The mapping f satisfying the relation (1.2) is called *harmonic*.

Let us define harmonic mappings and quasiconformal mappings between two smooth 2-dimensional surfaces $S_1 \subset \mathbb{R}^m$ and $S_2 \subset \mathbb{R}^n$. For every $a \in S_1$ let $X_a(x, y)$ be a *conformal mapping* between the unit disk and a neighborhood $U_a \subset S_1$ i.e. let x, y be *isothermal coordinates* in U_a. The mapping f of the surface S_1 into the surface S_2 is called *harmonic* if for every $a \in S_1$ $f \circ X_a : \mathbb{U} \to \mathbb{R}^n$ is harmonic in $\mathbb{U}$. Let $Y = Y_{f(a)}$ be isothermal coordinates in some neighborhood $V_{f(a)}$ in Y. It means that $g = (Y_{f(a)})^{-1} \circ f \circ X_a$ is $\rho-$harmonic, where $\rho(w) = |Y_u(w)|^2$; we also say that f is harmonic with respect to the metric on S_2 inherited from the Euclidean space $\mathbb{R}^n$. Let $k \in [0, 1)$ and let f be a homeomorphism between S_1 and S_2. Let $a \in S^1$ be arbitrary and let X_a be isothermal coordinates in U_a. Similarly let $Y_{f(a)}$ be isothermal coordinates in some neighborhood $V_{f(a)}$ in Y. If for every a the mapping $g = (Y_{f(a)})^{-1} \circ f \circ X_a$ satisfies the inequality $|g_{\bar{z}}| \leq k|g_z|$ in $X_a^{-1}(U_a)$, then f is said to be a k *quasiconformal (q.c.) mapping*.

The domain Ω (the surface S) is called $C^{l,\alpha}$ domain (surface) if the boundary $\partial\Omega$ (∂S) is a compact $C^{l,\alpha}$ 1-dimensional manifold (curve).

In this paper we continue to study the boundary behaviors of q.c. harmonic mappings between plane domains and Euclidean surfaces. Notice this important fact, the class of q.c. harmonic mappings contains conformal mappings (see the section below for boundary behaviors of conformal mappings).

Further developments of the method presented in [7] leads to Theorems 3.1 and 3.4: it is proved that every q.c. C^2 diffeomorphism w between two plane domains with smooth boundaries satisfying the inequality

$$|\Delta w| \leq M|\nabla w|^2 + N \tag{1.3}$$

is Lipschitz continuous. The inequality (1.3) we will call a *Poisson differential inequality*. These theorems imply corresponding results for q.c. harmonic mappings between smooth surfaces (Theorems 3.6 and 3.7). This extends the results of the authors [7] where instead of (1.3) is assumed that

$$|\Delta w| \leq M|w_z w_{\bar{z}}|. \tag{1.4}$$

For the background on the theory of q.c. harmonic mappings in the plane we refer to the papers [3–9, 11–16].

2 Conformal Parametrization

Proposition 2.1 (Kellogg and Warshawski see [10, 17, 18] and [19]) *Let $l \in \mathbb{N}$, $0 < \alpha \leq 1$. If Ω and D are Jordan domains having $C^{l,\alpha}$ boundaries and ω is a conformal mapping of Ω onto D, then $\omega^{(l)} \in C^\alpha(\overline{\Omega})$. In particular $\omega^{(l)}$ is bounded from above on $\mathbb{U}$.*

The following theorem can be viewed as an extension of Proposition 2.1 and of Riemann mapping theorem.

Theorem 2.2 [2, Theorem 3.1] *Suppose S is a surface with boundary, homeomorphic to a plane domain G bounded by k circles via a chart $\psi : \overline{G} \mapsto S$. Suppose the coefficients of the metric tensor of S can be defined in this chart by bounded measurable functions g_{ij} with $g_{11}g_{22} - g_{12}^2 \geq \lambda > 0$ in G. Then S admits a conformal representation $\tau \in H_1^2 \cap C^\alpha(\bar{B}, \bar{G})$, where B is a plane domain bounded by k circles and τ satisfies almost everywhere the conformality relations*

$$|\tau_x|^2 = |\tau_y|^2 \text{ and } \langle \tau_x, \tau_y \rangle = 0$$

(Here (x, y) denote the coordinates of points in B, and norms and products are taken with respect to the metric of S).

τ can be normalized by a three point condition, namely three points on one of the boundary curves of S can be made to correspond, respectively, to three given points on the outer boundary of B which can be taken as the unit circle, or by fixing the image of an interior point. Furthermore, concerning higher regularity, τ is as regular as S, i.e. if S is of class $C^{m,\alpha}(\bar{B})$ ($m \in \mathbb{N}$, $0 < \alpha < 1$) or in C^∞ then also $\tau \in C^{m,\alpha}(\bar{B})$ or $\tau \in C^\infty(\bar{B})$, respectively. In particular, if S is at least $C^{1,\alpha}$ then the conformality relations are satisfied everywhere, and τ is a diffeomorphism.

We will make use the following corollary of the previous theorem.

Corollary 2.3 *Let $X : \mathbb{U} \mapsto S$ be a conformal mapping between the unit disk and a $C^{2,\alpha}$ surface S. Then*

$$c := \min_{\{z:|z|\leq 1\}} |X_u(z)| = \min_{\{z:|z|\leq 1\}} |X_v(z)| > 0, \tag{2.1}$$

$$C := \max_{\{z:|z|\leq 1\}} |X_{uu}(z)| + |X_{uv}(z)| + |X_{vv}(z)| < \infty, \text{ and} \tag{2.2}$$

$$|\log |X_u(w)|^2{}_w| \leq M' < \infty. \tag{2.3}$$

Proof The first two inequalities follow directly from Theorem 2.2. For $\rho = \log |X_u(w)|^2$ we have

$$\rho_w = \frac{< X_{uu}, X_u > -i < X_{uv}, X_u >}{|X_u(w)|^2} = \frac{< X_{uu}, X_u > +i < X_{uu}, X_v >}{|X_u(w)|^2}.$$

Consequently:

$$|\rho_w| \leq 2\frac{|X_{uu}|}{|X_u|} \leq M' = \frac{C}{c}. \tag{2.4}$$

□

3 The Main Results

Firstly we are going to establish a local Lipschitz character of our mappings.

Theorem 3.1 (The main theorem) *Let f be a quasiconformal C^2 diffeomorphism from the plane domain Ω onto the plane domain G. Let $\gamma_\Omega \subset \partial\Omega$ and $\gamma_G = f(\gamma_\Omega) \subset$*

∂G be $C^{1,\alpha}$ respectively $C^{2,\alpha}$ Jordan arcs. If for some $\tau \in \gamma_\Omega$ there exist positive constants r, M and N such that

$$|\Delta f| \leq M|\nabla f|^2 + N, \quad z \in \Omega \cap D(\tau, r), \tag{3.1}$$

then f has bounded partial derivatives in $\Omega \cap D(\tau, r_\tau)$ for some $r_\tau < r$. In particular it is a Lipschitz mapping in $\Omega \cap D(\tau, r_\tau)$.

We need the following proposition.

Proposition 3.2 (Heinz–Bernstein, see [1]). *Let $s : \overline{\mathbb{U}} \to \mathbb{R}$ be a continuous function from the closed unit disc $\overline{\mathbb{U}}$ into the real line satisfying the conditions:*

(1) *s is C^2 on $\mathbb{U}$,*
(2) *$s_b(\theta) = s(e^{i\theta})$ is C^2 and*
(3) *$|\Delta s| \leq M_0|\nabla s|^2 + N_0$, on $\mathbb{U}$ for some constants M_0 and N_0.*

Then the function $|\nabla s|$ is bounded on $\mathbb{U}$.

Proof of Theorem 3.1 Let $r > 0$ be sufficiently small positive real number such that $\Delta = D(\tau, r) \cap \Omega$ is a Jordan domain with $C^{1,\alpha}$ boundary consisting of a circle arc $C(t_0, t_1)$ and an arc $\gamma_0[t_0, t_1] \subset \gamma$ containing τ. Take $D = f(\Delta)$. Let g be a conformal mapping of the unit disc onto Δ. Let $\tilde{f} = f \circ g$. Since $\Delta \tilde{f} = |g'|^2 \Delta f$ and $|\nabla \tilde{f}|^2 = |g'|^2 |\nabla f|^2$, we find that, $\tilde{f}$ satisfies the inequality (3.1) with $M_1 = M$ and $N_1 = N \cdot \inf_{|z|\leq 1} |g'(z)|^{-1}$. We will prove the theorem for $\tilde{f}$ and then apply Kellogg's theorem. For simplicity, we write f instead of $\tilde{f}$. Let J be a compact subset of γ_0 containing τ but not containing the points t_0 and t_1. Let $t \in J$ be arbitrary.

Step 1 (Local Construction). In this step we show that there are two Jordan domains D_1 and D_2 in D with $C^{2,\alpha}$ boundary such that

(i) $D_1 \subset D_2 \subset D$,
(ii) $\partial D \cap \partial D_2$ is a connected arc containing the point $w = f(t)$ in its interior,
(iii) $\emptyset \neq \overline{\partial D_2 \setminus \partial D_1} \subset D$.

Let H_1 be the Jordan domain bounded by the Jordan curve γ_1 which is composed by the following sequence of Jordan arcs: $\{y^{1/5} + (2-x)^{1/5} = 1,\ 1 \leq x \leq 2\}$; $\{(2-y)^{1/5} + (2-x)^{1/5} = 1,\ 1 \leq x \leq 2\}$; $[(1,2),(-1,2)]$; $\{(2-y)^{1/5} + (2+x)^{1/5} = 1, -2 \leq x \leq -1\}$;$\{y^{1/5} + (2+x)^{1/5} = 1, -2 \leq x \leq -1\}$ and $[(-1,0),(1,0)]$. Let H_2 be the Jordan domain bounded by the Jordan curve γ_2 which is composed by the following sequence of Jordan arcs: $\{y^{1/5} + (2-x)^{1/5} = 1,\ 1 \leq x \leq 2\}$; $[(2,1),(2,2)]$; $\{(3-y)^{1/5} + (2-x)^{1/5} = 1,\ 1 \leq x \leq 2\}$; $[(1,3),(-1,3)]$; $\{(3-y)^{1/5} + (2+x)^{1/5} = 1,\ -2 \leq x \leq -1\}$; $[(-2,2),(-2,1)]$; $\{y^{1/5} + (2+x)^{1/5} = 1,\ -2 \leq x \leq -1\}$ and $[(-1,0),(1,0)]$. Note that $H_1 \subset H_2 \subset [-2,2] \times [0,3]$, $\partial H_1 \cap \mathbb{R} = \partial H_2 \cap \mathbb{R} = [-1,1]$ and that $\partial H_1, \partial H_2 \in C^3$.

Let Γ be an orientation preserving arc-length parameterization of $\gamma = \partial D$ such that for $s_0 \in (0, \text{length}(\gamma))$ there holds: $\Gamma(s_0) = f(t)$. Let $D^* = \overline{\Gamma'(s_0)} D$, $b = \overline{\Gamma'(s_0)} f(t)$ and $\Gamma^* = \overline{\Gamma'(s_0)}\Gamma$. Then there exists $r > 0$ such that $(b, b + ir] \subset D^*$. Since $\gamma^* = \partial D^* \in C^{2,\alpha}$, it follows that, there exist

$x_0 > 0$, $\varepsilon > 0$, $y_0 \in (0, r/3)$, the $C^{2,\alpha}$ function $h : [-2x_0, 2x_0] \to \mathbb{R}$, $h(0) = 0$, and the domain $D_2^* \subset D^*$ such that:

(1) $\Gamma^*([s_0 - \varepsilon, s_0 + \varepsilon]) = \{b + (x, h(x)) : x \in [-2x_0, 2x_0]\}$,
(2) $D_2^* = \{b + (x, h(x) + y) : x \in [-2x_0, 2x_0],\ y \in (0, 3y_0]\}$.

Let $\Upsilon : [-2, 2] \times [0, 3] \to D_2^*$ be the mapping defined by:

$$\Upsilon(x, y) = b + (xx_0, h(xx_0) + yy_0).$$

Then Υ is a $C^{2,\alpha}$ diffeomophism.

Take $D_i = \Gamma'(s_0) \cdot \Upsilon(H_i)$, $i = 1, 2$. Obviously $D_1 \subset D_2 \subset D$ and D_1 and D_2 have $C^{2,\alpha}$ boundary. Observe that $f(t) = \Gamma'(s_0)\overline{\Gamma'(s_0)}f(t) = \Gamma'(s_0)\Upsilon(0) \in \Gamma'(s_0)\Upsilon([-1, 1]) = \partial D_1 \cap \partial D_2$.

Step 2 (Application of Heinz–Bernstein theorem). Let ϕ be a conformal mapping of D_2 onto H such that $\phi^{-1}(\infty) \in \partial D_2 \setminus \partial D_1$. Let $\Omega_1 = \phi(D_1)$. Then there exist real numbers a, b, c, d such that $a < c < d < b$, $[a, b] = \partial\Omega_1 \cap \mathbb{R}$ and $l = \phi^{-1}(\partial\Omega_1 \setminus [c, d]) \subset D$. Let $U_1 = f^{-1}(D_1)$ and η be a conformal mapping between the unit disc and the domain U_1. Then the mapping $\hat{f} = \phi \circ f \circ \eta$ is a C^2 diffeomorphism of the unit disc onto the domain Ω_1 such that:

(a) $\hat{f}$ is continuous on the boundary $\mathbb{T} = \partial\mathbb{U}$ (it is q.c.) and
(b) $\hat{f}$ is C^2 on the set $T_1 = \hat{f}^{-1}(\partial\Omega_1 \setminus (c, d))$.

Let $s := \operatorname{Im} \hat{f}$. First, note that (a) implies that s is continuous on $\mathbb{T} = \partial\mathbb{U}$. On other hand, as $\hat{f} \in C^2$, s satisfies the condition:

(1) $s \in C^2(\mathbb{U})$.

From (b) we obtain that s is C^2 on the set $T_1 = \hat{f}^{-1}(\partial\Omega \setminus (c, d))$. Furthermore, $s = 0$ on $T_2 = \hat{f}^{-1}(a, b)$; and therefore s is C^2 on $T_2 = \hat{f}^{-1}(a, b)$. Hence:

(2) s is C^2 on $\mathbb{T} = T_1 \cup T_2$. In other words, the function $s_b : \mathbb{R} \to \mathbb{R}$ defined by $s_b(\theta) = s(e^{i\theta})$ is C^2 in $\mathbb{R}$.

In order to apply the interior estimate, we have to prove that

(3) $|\Delta s(z)| \le M_0|\nabla s(z)|^2 + N_0$, $z \in \mathbb{U}$, where M_0 and N_0 are constants.

To continue we need the following lemma:

Lemma 3.3 *If $f = u + iv$ is a q.c. mapping satisfying Poisson differential inequality, then u and v satisfy the Poisson differential inequality.*

Proof Let

$$A := |\nabla u|^2 = 2(|u_z|^2 + |u_{\bar{z}}|^2) = \frac{1}{2}(|f_z + \overline{f_{\bar{z}}}|^2 + |f_{\bar{z}} + \overline{f_z}|^2)$$

and

$$B := |\nabla v|^2 = 2(|v_z|^2 + |v_{\bar{z}}|^2) = \frac{1}{2}(|f_z - \overline{f_{\bar{z}}}|^2 + |f_{\bar{z}} - \overline{f_z}|^2).$$

Then

$$\frac{A}{B} = \frac{|1+\mu|^2}{|1-\mu|^2}$$

where $\mu = \overline{f_{\bar z}}/f_z$. Since $|\mu| \le k$

$$\frac{(1-k)^2}{(1+k)^2} \le \frac{A}{B} \le \frac{(1+k)^2}{(1-k)^2}. \tag{3.2}$$

As

$$|\Delta f| = |\Delta u + i\Delta v| \le M|\nabla f|^2 + N = M(|\nabla u|^2 + |\nabla v|^2) + N,$$

the relation (3.2) yields

$$|\Delta u| \le M\frac{(1+k)^2}{(1-k)^2}|\nabla u|^2 + N$$

and

$$|\Delta v| \le M\frac{(1+k)^2}{(1-k)^2}|\nabla v|^2 + N.$$

Since $\hat f = \phi \circ f \circ \eta$, we obtain

$$\partial \hat f = \phi' \partial f \eta', \quad \bar\partial \hat f = \phi' \bar\partial f \eta' \tag{3.3}$$

and

$$\partial\bar\partial \hat f = \frac{1}{4}\Delta \hat f = \frac{1}{4}\Delta(\phi \circ f) \cdot |\eta'|^2 = (\phi''\partial f \cdot \bar\partial f + \phi'\partial\bar\partial f)|\eta'|^2. \tag{3.4}$$

Now combining (3.1), (3.3) and (3.4) we obtain

$$\begin{aligned}
|\Delta \hat f| &\le 4\frac{|\phi''|}{|\phi'|^2}|\partial \hat f||\bar\partial \hat f| + |\phi'||\Delta f||\eta'|^2 \\
&\le 4\frac{|\phi''|}{|\phi'|^2}|\partial \hat f||\bar\partial \hat f| + |\phi'|\left(M|\nabla f|^2 + N\right)|\eta'|^2 \\
&\le \frac{|\phi''|}{|\phi'|^2}|\nabla \hat f|^2 + M|\nabla \hat f|^2 \cdot \frac{1}{|\phi'|} + N|\phi'||\eta'|^2 \\
&= \left(\frac{|\phi''|}{|\phi'|^2} + \frac{M}{|\phi'|}\right)|\nabla \hat f|^2 + N|\phi'||\eta'|^2.
\end{aligned}$$

As $\hat f$ is a k–q.c. mapping using Lemma 3.3 we have

$$|\Delta s| \le \frac{(1+k)^2}{(1-k)^2}\left(\frac{|\phi''|}{|\phi'|^2} + \frac{M}{|\phi'|}\right) \cdot |\nabla s|^2 + N|\phi'||\eta'|^2. \tag{3.5}$$

Proposition 2.1 implies that the function $|\eta'|$ is bounded from above by a constant C_1, the function $|\phi'|$ is bounded from below and above by positive constants C_2 and C_3 respectively and the function $|\phi''|$ is bounded from above by a constant C_4. Hence

$$|\Delta s| \le M_0 |\nabla s|^2 + N_0,$$

where

$$M_0 = \frac{(1+k)^2}{(1-k)^2}\left(\frac{C_4}{C_3^2} + \frac{M}{C_3}\right) \text{ and } N_0 = C_2 C_1^2 N.$$

Proposition 3.2 implies that, the function $|\nabla s|$ is bounded by a constant b_t. Since $\hat{f}$ is a $k-$q.c. mapping, we have

$$(1-k)|\partial \hat{f}| \le |\partial \hat{f} - \overline{\bar{\partial} \hat{f}}| \le 2|s_z| \le \sqrt{2} b_t.$$

Finally,

$$|\partial \hat{f}| + |\bar{\partial} \hat{f}| \le \sqrt{2}\frac{1+k}{1-k} b_t.$$

Since the mapping η is conformal and maps the circle arc $T = (\phi \circ f \circ \eta)^{-1}(a, b)$ onto the circle arc $(\phi \circ f)^{-1}(a, b)$, it follows that, it can be conformally extended across the arc $T' = (\phi \circ f \circ \eta)^{-1}[c, d]$. Hence, there exists a constant A such that $|\eta'(z)| \ge 2A$ on T'. It follows that there exists $r \in (0, 1)$ such that $|\eta'(z)| \ge A$ in $\{\rho z : z \in T', r \le \rho \le 1\}$. It follows from the Proposition 2.1, that the conformal mapping ϕ and its inverse have the C^1 extension to the boundary. Therefore there exists a positive constant B such that $|\phi'(z)| \ge B$ on some neighborhood of $\phi^{-1}[c, d]$ with respect to D. Thus, the mapping $f = \phi^{-1} \circ \hat{f} \circ \eta^{-1}$ has bounded derivative in some neighborhood of the set $\eta(T')$, on which it is bounded by the constant

$$C = \sqrt{2}\frac{1+k}{1-k}\frac{b_t}{AB}.$$

Then

$$|\partial f(z)| + |\bar{\partial} f(z)| \le C_0 \text{ for all } z \in \mathbb{U} \text{ near the arc } T = \eta(T').$$

□

Theorem 3.4 *Let f be a quasiconformal C^2 diffeomorphism from the plane domain Ω with $C^{1,\alpha}$ compact boundary onto the plane domain G with $C^{2,\alpha}$ compact boundary. If there exist constants M and N such that*

$$|\Delta f| \le M|\nabla f|^2 + N, \quad z \in \Omega, \tag{3.6}$$

then f has bounded partial derivatives in Ω. In particular it is a Lipschitz mapping in Ω.

Proof According to the Theorem 3.1 for every $t \in \partial\Omega$ there exists $r_t > 0$ such that f has bounded partial derivatives in $\Omega \cap D(t, r_t)$. Since $\partial\Omega$ is a compact set it follows that there exist $t_1, \dots, t_m$ such that $\partial\Omega \subset \bigcup_{i=1}^m D(t_i, r_{t_i})$. It follows that f has bounded partial derivatives in $\Omega \cap \bigcup_{i=1}^m D(t_i, r_{t_i})$. Since f is a diffeomorphism in Ω, we obtain that f has bounded derivatives in the compact set $\Omega \setminus \bigcup_{i=1}^{m} D(t_i, r_{t_i})$. The conclusion of the theorem now easily follows. □

Corollary 3.5 *Let Ω be a plane domain with $C^{1,\alpha}$ compact boundary and G be a plane domain with $C^{2,\alpha}$ compact boundary. If $w = f(z) : \Omega \mapsto G$ is a quasiconformal solution of the equation*

$$\begin{aligned} &\alpha w_{xx} + 2\beta w_{xy} + \gamma w_{yy} + a_1(z) w_x^2 + b_1(z) w_x w_y + c_1(z) w_y^2 \\ &\quad + a(z) w_x + b(z) w_y + c(z) w + d(z) = 0, \end{aligned} \tag{3.7}$$

such that $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha > 0$, $\alpha\gamma - \beta^2 > 0$, $a, b, c, d, a_1, b_1, c_1 \in C(\overline{\Omega})$, then f is Lipschitz.

Proof Since the partial differential equation (PDE) (3.7) is elliptic, we can choose coordinates $x = \alpha_1 u + \beta_1 v$, $y = \beta_1 u + \gamma_1 v$ such that (3.7) becomes

$$\begin{aligned} &w_{uu} + w_{vv} + a_1'(u,v) w_u^2 + b_1'(u,v) w_u w_v + c_1'(u,v) w_v^2 \\ &\quad + a'(u,v) w_u + b'(u,v) w_v + c'(u,v) w + d'(u,v) = 0, \ \ (u,v) \in \Omega'. \end{aligned} \tag{3.8}$$

For $e \in C(\overline{\Omega'})$ let $|e| = \max\{|e(u,v)| : (u,v) \in \overline{\Omega'}\}$. Using (3.8) and the inequality $2|t| \leq |t|^2 + 1$ we obtain

$$\begin{aligned} |\Delta w| &\leq \left(\frac{|a'|}{2} + \frac{|b'|}{2}\right)(|\nabla w|^2 + 1) + \left(\max\{|a_1'|, |c_1'|\} + \frac{|b_1'|}{2}\right)|\nabla w|^2 + |c'||w| + |d'| \\ &= M|\nabla w|^2 + N, \end{aligned}$$

where

$$M = (|a'| + |b'|)/2 + \max\{|a_1'|, |c_1'|\} + \frac{|b_1'|}{2}$$

and

$$N = \frac{|a'| + |b'|}{2} + |c'||w| + |d'|.$$

The conclusion now follows from Theorem 3.4. □

By $d = d_k$ we denote Euclidean distance in Euclidean space $\mathbb{R}^k$.

Theorem 3.6 *We call a $C^{2,\alpha}$ surface S disk-like surface if it is homeomorphic to the unit disk, and if its boundary is a $C^{2,\alpha}$ curve. If f is a quasiconformal harmonic mapping between two $C^{2,\alpha}$ disk-like surfaces S_1 and S_2, then it is a Lipschitz mapping i.e. there exists a constant C such that*

$$d(f(x), f(y)) \leq C d(x, y), \ \textit{for all } x, y \in S_1.$$

Proof Let f be a harmonic q.c. mapping between disk-like surfaces S_1 and S_2. Let $X : \mathbb{U} \mapsto S_1$ and $Y : \mathbb{U} \mapsto S_2$ be conformal mappings. Let us consider the mapping $g = Y^{-1} \circ f \circ X$ of the unit disk onto itself. Since $f(X(z)) = Y(g(z))$, it follows that

$$|f \circ X_x|^2 + |f \circ X_y|^2 = |Y_u|^2(|g_x|^2 + |g_y|^2).$$

Hence

$$E[f \circ X] = E_Y[g] = \int\int_{\Omega} |Y_u|^2(|g_x|^2 + |g_y|^2)dxdy. \tag{3.9}$$

If we denote $\rho(w) = |Y_u(w)|^2$, then the stationary points of the energy integral $E_Y[g]$ satisfies the Euler–Lagrange equation

$$g_{z\bar{z}} + (\log \rho)_w \circ g g_z g_{\bar{z}} = 0. \tag{3.10}$$

Consequently $f \circ X$ is harmonic if and only if g is $\rho-$harmonic i.e. the mapping satisfying the relation (3.10). According to the Corollary 2.3, the mapping g satisfies the conditions of Theorem 3.4. Namely as $|(\log \rho)_w| \leq M$ and $|g_z g_{\bar{z}}| \leq 1/2(|g_z|^2 + |g_{\bar{z}}|^2)$ we can simply take $M = M'/2$, and $N = 0$. Theorem 3.4 yields that g is Lipschitz. By Theorem 2.2 it follows that X and Y are bi-Lipschitz mappings. f is Lipschitz as a composition of Lipschitz mappings. □

Using Theorem 3.6 we obtain the theorem:

Theorem 3.7 *If f is a quasiconformal harmonic mapping between $C^{2,\alpha}$ surfaces S_1 and S_2, with $C^{2,\alpha}$ compact boundaries then it is a Lipschitz mapping i.e. there exists a constant C such that $d(f(x), f(y)) \leq Cd(x, y)$, for all $x, y \in S_1$.*

3.1 A Question

Recently in [6] is proved that, every quasiconformal harmonic mapping between two Jordan domains $\Omega_1 \in C^{1,\alpha}$ and $\Omega_2 \in C^{2,\alpha}$ is bi-Lipschitz, and this can be extended directly to all $C^{j,\alpha}$, $j = 1, 2$ plane domains. On the other hand, this result has been extended in [5] to the $C^{2,\alpha}$ surface with approximately analytic metrics. The question arises, whether the previous statement can be extended to $C^{1,\alpha}$ surfaces?

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