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On the quasi-isometries of harmonic quasiconformal mappings

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Abstract

We prove versions of the Ahlfors–Schwarz lemma for quasiconformal euclidean harmonic functions and harmonic mappings with respect to the Poincaré metric.

Keywords: Quasiconformal harmonic mappings; Hyperbolic density; Poincaré metric; Gaussian curvature

1. Introduction

In order to discuss the subject it is convenient to give a few comments about the notation.

Let $\mathbb{U} = \{z \in \mathbb{C}\colon |z| < 1\}$ denote the unit disc.

Occasionally, we write qc, qr, ρ-qch, and ρ-qrh instead of quasiconformal, quasiregular, ρ-harmonic quasiconformal and ρ-harmonic quasiregular, respectively and e-qch, h-qch instead of euclidean and hyperbolic harmonic quasiconformal. Basic definitions will be given in Section 2.

The Schwarz lemma attracted a lot of attention and found numerous applications in geometric function theory.

It seems that investigations concerning the Schwarz lemma have been primarily concerned with the following question:

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Question A. Which properties of holomorphic functions and the *Poincaré* metric are essential for validity of the Schwarz lemma?

For our purpose the following are relevant.

Ahlfors lemma. *If $\rho > 0$ is a C^2 function (metric density) on $\mathbb{U}$ and the Gaussian curvature satisfies $K_\rho \leqslant -1$, then $\rho \leqslant \lambda$.*

Sometimes we refer to this result as the Ahlfors–Schwarz lemma.

In [12], Yau mentioned that in order to draw a useful conclusion in the case of harmonic mappings between Riemannian manifolds, one has to assume the mapping is quasiconformal. According to our knowledge, Wan [11] was the first one who showed a result in a special situation concerning Yau's suggestion:

W-lemma *(Wan). Every hyperbolic harmonic quasiconformal diffeomorphism from $\mathbb{U}$ onto itself is a quasi-isometry of the Poincaré disc.*

In particular, the method of the proof is interesting. It provides at least a partial motivation to study this approach and raises the following question:

Question B. Which properties of hyperbolic harmonic mappings and the *Poincaré* metric are essential for validity of a version of the Ahlfors–Schwarz lemma (and in particular, Wan's theorem)?

In [6], the second author proved an inequality of opposite type of the Ahlfors–Schwarz lemma:

Lemma A. *If $H > 0$ is a C^2 metric density on $\mathbb{U}$ for which the Gaussian curvature satisfies $K_H \geqslant -1$ and if $H(z)$ tends to $+\infty$, when $|z|$ tends to 1_-, then $\lambda \leqslant H$.*

We will use this lemma together with the Ahlfors–Schwarz lemma in Section 3.

We prove an analogue of the W-lemma holds for quasiconformal euclidean harmonic mappings and we generalize it to quasiregular harmonic mappings with respect to the metric ρ, whose curvature is bounded from above by a negative constant.

It is interesting that we have a similar estimate of the hyperbolic distance for qc euclidean harmonic mappings and harmonic mappings with respect to the Poincaré metric, which are different in many respects.

We now give more details concerning our results.

In Section 2, which is mainly expository, we discuss basic definitions, background and known results including the Ahlfors–Schwarz lemma (non-analytic form of the Schwarz lemma).

Let f be a K-qc euclidean harmonic diffeomorphism from a domain D onto itself. We show that f is a $(1/K, K)$ quasi-isometry with respect to the Poincaré distance in the case where D is the disc or the upper-half plane. We refer to these results as the unit disc and the half plane euclidean-qch versions, respectively.

The proofs of these cases cannot be transferred to one another using conformal mappings because the euclidean metric is not invariant under them.

Theorem *(The half plane and the unit disc e-qch versions). Let f be a K-qc euclidean harmonic diffeomorphism from the upper half plane H (or the unit disc) onto itself. Then f is a $(1/K, K)$ quasi-isometry with respect to the Poincaré distance.*

It is interesting that we use completely different techniques for the disc and the half plane. In the case of the unit disc we use a curvature estimate (see below). In the case of the upper half plane, the following known fact plays an important role:

Lemma B. *Let f be an euclidean harmonic 1-1 mapping of the upper half-plane $\mathbb{H}$ onto itself, continuous on $\bar{\mathbb{H}}$, normalized by $f(\infty)=\infty$ and $v=\operatorname{Im} f$. Then $v(z)=c\operatorname{Im} z$, where c is a positive constant. In particular, v has bounded partial derivatives on $\mathbb{H}$.*

This lemma is a corollary of the Herglotz representation of the positive harmonic function v (see for example [2]).

For information regarding the quasi-isometries, with respect to the hyperbolic metric for e-qch mappings with general codomains, see [9].

In Section 5, we extend Wan's result to qr ρ-harmonic mappings (see Theorem 5.1):

Theorem *(ρ-qrh version). Let R be a hyperbolic surface with the Poincaré metric density λ, S a hyperbolic surface with metric density ρ and let the Gaussian curvature of the metric $ds^2=\rho(w)|dw|^2$ be uniformly bounded from above on S by the negative constant $-a$. Then any ρ-harmonic k-quasiregular map f from R into S decreases distances up to a constant depending only on a and k.*

The basic properties of ρ-harmonic functions will be briefly discussed in Section 2.4.

Roughly speaking, a proof of the above result can be based on an application of the uniformization theorem with the fact that $\rho_0=a(1-k^2)\rho\circ f|f_z|^2$ is an ultrahyperbolic metric density.

2. Definitions and Ahlfors–Schwarz lemma

2.1. Hyperbolic distance and Schwarz lemma

Using the conformal automorphisms $\phi_a(z)=\frac{z-a}{1-\bar{a}z}$, $a\in\mathbb{U}$, of $\mathbb{U}$, one can define the pseudo-hyperbolic distance on $\mathbb{U}$ by

$$\delta(a,b)=\left|\phi_a(b)\right|,\quad a,b\in\mathbb{U}.$$

The hyperbolic metric on the unit disc $\mathbb{U}$ is $\lambda|dz|^2$, where

$$\lambda(z)=\left(\frac{2}{1-|z|^2}\right)^2.$$

We say that λ is the hyperbolic metric density. The hyperbolic distance on the unit disc $\mathbb{U}$ is

$$d_\lambda(z,\omega)=\ln\frac{1+\delta(z,\omega)}{1-\delta(z,\omega)}=\ln\frac{1+|\frac{z-\omega}{1-z\bar{\omega}}|}{1-|\frac{z-\omega}{1-z\bar{\omega}}|}.$$

We also use the notation d_h instead of d_λ.

The classical Schwarz lemma states: If $f:\mathbb{U}\to\mathbb{U}$ is an analytic function and if $f(0)=0$, then $|f(z)|\leqslant|z|$ and $|f'(0)|\leqslant 1$. Equality $|f(z)|=|z|$, with $z\neq 0$, or $|f'(0)|=1$ can occur only for $f(z)=e^{i\alpha}z$, where α is a real constant.

It was noted by Pick that the result can be expressed in invariant form. We refer to the following result as the Schwarz–Pick lemma.

Theorem A *(Schwarz–Pick lemma). Let F be an analytic function from the unit disc into itself. Then F does not increase the corresponding hyperbolic (pseudo-hyperbolic) distances.*

2.2. Curvature

A Riemannian metric given by the fundamental form

$$ds^2=\rho\big(dx^2+dy^2\big)=\rho|dz|^2$$

or $ds=\sqrt{\rho}|dz|$, $\rho>0$, is conformal to the euclidean metric. We call ρ a metric density (scale) and denote by d_ρ the corresponding distance.

If $\rho>0$ is a C^2 function on $\mathbb{U}$, the Gaussian curvature of a Riemannian metric with density ρ on $\mathbb{U}$ is expressed by the formula

$$K=K_\rho=-\frac{1}{2\rho}\Delta\ln\rho.$$

We also write $K(\rho)$ instead of K_ρ. If $s>0$ is a constant, it is clear that $K(s\rho)=s^{-1}K(\rho)$.

2.3. Ahlfors–Schwarz lemma

A metric $\rho|dz|^2$, $\rho\geqslant 0$, is said to be ultrahyperbolic in a region $\Omega\subset\mathbb{C}$ if it has the following properties:

(a) ρ is upper semicontinuous; and
(b) at every z_0 with $\rho(z_0)>0$ there exists a supporting metric density ρ_0, of class C^2 in a neighborhood V of z_0, such that $\rho_0\leqslant\rho$ and $K_{\rho_0}\leqslant -1$ in V, while $\rho_0(z_0)=\rho(z_0)$.

If a metric $\rho|dz|^2$ is ultrahyperbolic in a region $\Omega\subset\mathbb{C}$ we say that ρ is an ultrahyperbolic metric density.

Ahlfors (see [1]) proved a stronger version of the Schwarz–Pick lemma and of the Ahlfors–Schwarz lemma.

Theorem B *(Ahlfors–Schwarz lemma). Suppose ρ is an ultrahyperbolic metric on the unit disc $\mathbb{U}$. Then $\rho\leqslant\lambda$.*

Sometimes we refer to this result as the Ahlfors–Schwarz lemma or the non-analytic form of the Schwarz lemma. If we wish to be more specific, we refer to this result as the Ahlfors ultrahyperbolic lemma.

Now, we can state Theorem B in the following form: If ρ is a metric density on $\mathbb{U}$ such that $K_\rho(z)\leqslant -a$, for some $a>0$, then the metric $a\rho$ is ultrahyperbolic and therefore $a\rho\leqslant\lambda$.

The notation of an ultrahyperbolic metric makes sense and the theorem remains valid if Ω is replaced by a Riemann surface.

In a plane region Ω whose complement has at least two points, there exists a unique maximal ultrahyperbolic metric and this metric has constant curvature of -1.

The maximal metric density is called the *Poincaré* metric (density) in Ω and we denote it by λ_Ω. It is maximal in the sense that every ultrahyperbolic metric density ρ satisfies $\rho \leqslant \lambda_\Omega$ throughout Ω.

Ultrahyperbolic metrics (without the name) were introduced by Ahlfors. They found many applications in the theory of several complex variables.

2.4. *Definition of harmonic quasiregular functions*

Let R and S be two surfaces. Let $\sigma(z)|dz|^2$ and $\rho(w)|dw|^2$ be metrics with respect to the isothermal coordinate charts on R and S, respectively, and let f be a C^2-map from R to S.

It is convenient to use the notation in local coordinates: $df = p\,dz + q\,d\bar{z}$, where $p = f_z$ and $q = f_{\bar{z}}$. We also introduce the complex (Beltrami) dilatation

$$\mu_f = Belt[f] = \frac{q}{p},$$

where it is defined.

We say that a C^2-map f from R to S is ρ-harmonic (harmonic with respect to the metric density ρ or, shortly, harmonic) if f satisfies the following equation:

$$f_{z\bar{z}} + (\log \rho)_w \circ f pq = 0.$$

For basic properties of harmonic maps and for further information we refer to Jost [3] and Schoen and Yau [10].

Note that if R and S are domains in the complex plane and if σ and ρ are the euclidean metric densities (that is $\sigma = \rho = 1$), then f is euclidean harmonic.

If $f : \mathbb{U} \to \mathbb{U}$ is a λ-harmonic mapping, we call f a hyperbolic harmonic or a harmonic mapping with respect to the Poincaré metric.

Let R and S be two Riemann surfaces and $f : R \to S$ be a C^2-mapping. If P is a point on R, $\tilde{P} = f(P) \in S$, ϕ a local parameter on R defined near P and ψ a local parameter on S defined near $\tilde{P}$, then the map $w = h(z)$ defined by $h = \psi \circ f \circ \phi^{-1}|_V$ (V is a sufficiently small neighborhood of P) is called a local representation of f at P. The map f is called k-quasiregular if there is a constant $k \in (0, 1)$ such that for every representation h, at every point of R, $|h_{\bar{z}}| \leqslant k|h_z|$.

If a k-qr mapping is one-to-one, we call it a k-qc mapping. Also, if f is a k-qc mapping, we use the notation $K = \frac{1+k}{1-k}$, and we also write that f is K-qc.

3. Euclidean harmonic functions

First, we need to introduce some notation. We write

$$L_f = L_f(z) = \left|f_z(z)\right| + \left|f_{\bar{z}}(z)\right| \quad \text{and} \quad l_f = l_f(z) = \left|f_z(z)\right| - \left|f_{\bar{z}}(z)\right|,$$

if $f_z(z)$ and $f_{\bar{z}}(z)$ exist.

The following statements are useful in applications:

3A If f is a C^1 mapping of $\mathbb{U}$ into itself and

$$\frac{L_f(z)}{1 - |f(z)|^2} \leqslant c_1 \frac{1}{1 - |z|^2}, \quad z \in \mathbb{U}, \tag{3.1}$$

then $d_h(f(z_1), f(z_2)) \leqslant c_1 d_h(z_1, z_2)$.

3B If f is a C^1 mapping of $\mathbb{U}$ onto itself and

$$\frac{l_f(z)}{1-|f(z)|^2} \geqslant c_2 \frac{1}{1-|z|^2}, \quad z \in \mathbb{U}, \tag{3.2}$$

then $d_h(f(z_1), f(z_2)) \geqslant c_2 d_h(z_1, z_2)$.

The proofs are straightforward. Note that in the proof of 3A it is convenient to consider the hyperbolic geodesic joining z_1 and z_2 and in the proof of 3B the hyperbolic geodesic joining $f(z_1)$ and $f(z_2)$.

Proposition 3.1 *(The unit disc euclidean-qch version). Let f be a k-quasiconformal euclidean harmonic mapping from the unit disc $\mathbb{U}$ into itself. Then for all $z \in \mathbb{U}$ we have*

$$\left|f_z(z)\right| \leqslant \frac{1}{1-k} \frac{1-|f(z)|^2}{1-|z|^2}.$$

Notice that as a corollary we get $(1-|z|^2)L_f(z) \leqslant 4K d_f(z)$, where $d_f(z) = \operatorname{dist}(f(z), \partial\mathbb{U})$.

Proof. Let us define $\sigma(z) = (1-k)^2 \lambda(f(z))|f_z(z)|^2$, $z \in \mathbb{U}$. Since f is harmonic in $\mathbb{U}$, i.e. $f_{z\bar{z}}(z) = 0$, $z \in \mathbb{U}$, then f_z is holomorphic in $\mathbb{U}$. By Lewy's theorem f_z does not vanish and hence the mapping $z \mapsto \log|f_z(z)|$ is harmonic in $\mathbb{U}$. Therefore, $(\Delta \log \sigma)(z) = (\Delta \log(\lambda \circ f))(z)$, for all $z \in \mathbb{U}$. A straightforward calculation gives

$$\begin{aligned}
(\Delta \log \sigma)(z) &= \big(\Delta \log(\lambda \circ f)\big)(z) = 4\big(\log(\lambda \circ f)\big)_{z\bar{z}}(z) \\
&= \frac{8|f_z(z)|^2}{(1-|f(z)|^2)^2}\left(1 + \left|\mu(z)\right|^2 + 2\operatorname{Re}\left(\frac{(f(z))^2 \overline{f_z(z)}\,\overline{f_{\bar{z}}(z)}}{|f_z(z)|^2}\right)\right) \\
&= \frac{2\sigma(z)}{(1-k)^2}\left(1 + \left|\mu(z)\right|^2 + 2\operatorname{Re}\left(\frac{(f(z))^2 \overline{f_z(z)}\,\overline{f_{\bar{z}}(z)}}{|f_z(z)|^2}\right)\right).
\end{aligned}$$

Hence, the Gaussian curvature of the conformal metric $ds^2 = \sigma(z)|dz|^2$ satisfies

$$K(\sigma)(z) = -\frac{1}{(1-k)^2}\left(1 + \left|\mu(z)\right|^2 + 2\operatorname{Re}\left(\frac{(f(z))^2 \overline{f_z(z)}\,\overline{f_{\bar{z}}(z)}}{|f_z(z)|^2}\right)\right) \tag{3.3}$$

for all $z \in \mathbb{U}$. On the other hand we have

$$\left|\operatorname{Re}\left(\frac{(f(z))^2 \overline{f_z(z)}\,\overline{f_{\bar{z}}(z)}}{|f_z(z)|^2}\right)\right| \leqslant \left|\frac{(f(z))^2 \overline{f_z(z)}\,\overline{f_{\bar{z}}(z)}}{|f_z(z)|^2}\right| \leqslant \left|\mu(z)\right|, \tag{3.4}$$

so we obtain

$$\operatorname{Re}\left(\frac{(f(z))^2 \overline{f_z(z)}\,\overline{f_{\bar{z}}(z)}}{|f_z(z)|^2}\right) \geqslant -\left|\mu(z)\right|.$$

Therefore,

$$K(\sigma)(z) \leqslant -\frac{1}{(1-k)^2}\left(1 + \left|\mu(z)\right|^2 - 2\left|\mu(z)\right|\right) = -\frac{(1-|\mu(z)|)^2}{(1-k)^2} \leqslant -1,$$

and hence, using the Ahlfors–Schwarz lemma, we get $\sigma(z) \leqslant \lambda(z)$, $z \in \mathbb{U}$, or equivalently

$$(1-k)^2 \lambda\big(f(z)\big)\left|f_z(z)\right|^2 \leqslant \lambda(z) \tag{3.5}$$

for all $z \in \mathbb{U}$. Now, the claim follows easily from (3.5). □

Theorem 3.1. *Let f be a k-quasiconformal euclidean harmonic mapping from the unit disc $\mathbb{U}$ into itself. Then for any two points z_1 and z_2 in $\mathbb{U}$ we have*

$$d_h\big(f(z_1), f(z_2)\big) \leqslant \frac{1+k}{1-k} d_h(z_1, z_2),$$

where d_h is the hyperbolic distance function induced by the hyperbolic metric in $\mathbb{U}$.

Note that this statement follows from Proposition 3.1 and 3A.

Notice that, in order to get the opposite inequality in Proposition 3.1, we need to assume that f is onto.

Theorem 3.2. *Let f be a k-quasiconformal euclidean harmonic mapping from the unit disc $\mathbb{U}$ onto itself. Then for all $z \in \mathbb{U}$ we have*

$$\big|f_z(z)\big| \geqslant \frac{1}{1+k} \frac{1-|f(z)|^2}{1-|z|^2}$$

and $d_h(f(z_1), f(z_2)) \geqslant \frac{1-k}{1+k} d_h(z_1, z_2)$.

Proof. By (3.3) and (3.4)

$$K(\sigma)(z) \geqslant -\frac{1}{(1-k)^2}\big(1+\big|\mu(z)\big|^2 + 2\big|\mu(z)\big|\big) = -\frac{(1+|\mu(z)|)^2}{(1-k)^2} \geqslant -K^2.$$

In [8], it has been proved that there is a constant $c > 0$ such that $|f_z| \geqslant c$ on $\mathbb{U}$. Hence, σ tends to $+\infty$, when $|z|$ tends to 1_-. Thus, by Lemma A, $(1+k)^2\lambda(f(z))|f_z(z)|^2 \geqslant \lambda(z)$ and therefore, since $l_f \geqslant (1-k)|f_z(z)|$, we have $K^2\lambda(f(z))l_f^2 \geqslant \lambda$, i.e. $K\hat{\lambda}(f(z))l_f \geqslant \hat{\lambda}$, where $\hat{\lambda} = \sqrt{\lambda}$.

Now, an application of 3B immediately yields $d_h(f(z_1), f(z_2)) \geqslant \frac{1-k}{1+k} d_h(z_1, z_2)$. □

4. The half plane

For $a \in \mathbb{C}$ and $r > 0$ we define $B(a; r) = \{z\colon |z-a| < r\}$. In particular, we write U_r instead of $B(0; r)$.

Theorem 4.1 *(The half plane euclidean-qch version). Let f be K-qc euclidean harmonic diffeomorphism from $\mathbb{H}$ onto itself. Then f is $(1/K, K)$ quasi-isometry with respect to the Poincaré distance.*

We first show that, by precomposition with a linear fractional transformation, we can reduce the proof to the case $f(\infty) = \infty$. If $f(\infty) \neq \infty$, there is a real number a such that $f(a) = \infty$. On the other hand, there is a conformal automorphism A of $\mathbb{H}$ such that $A(\infty) = a$. Since A is an isometry of $\mathbb{H}$ onto itself and $f \circ A$ is a K-qc euclidean harmonic diffeomorphism from $\mathbb{H}$ onto itself, the proof is reduced to the case $f(\infty) = \infty$.

It is well known that f has a continuous extension to $\bar{\mathbb{H}}$ (see for example [5]).

Hence, by Lemma B, $f = u + ic\operatorname{Im} z$, where c is a positive constant. Using the linear mapping B, defined by $B(w) = w/c$, and a similar consideration as the above, we can reduce the proof to the case $c = 1$. Therefore we can write f in the form $f = u + i\operatorname{Im} z = \frac{1}{2}(F(z) + z + \overline{F(z) - z})$, where F is a holomorphic function in $\mathbb{H}$. Hence,

$$\mu_f(z) = \frac{F'(z)-1}{F'(z)+1} \quad \text{and} \quad F'(z) = \frac{1+\mu_f(z)}{1-\mu_f(z)}, \quad z \in \mathbb{H}.$$

Define $w = S(\zeta) = \frac{1+\zeta}{1-\zeta}$. Then, $S(U_k) = B_k = B(a_k; R_k)$, where $a_k = \frac{1}{2}(K + 1/K)$ and $R_k = \frac{1}{2}(K - 1/K)$.

Since f is k-qc, then $\mu_f(z) \in U_k$ and therefore $F'(z) \in B_k$, for $z \in \mathbb{H}$. This yields, first,

$$K + 1 \geqslant \left|F'(z) + 1\right| \geqslant 1 + 1/K, \qquad K - 1 \geqslant \left|F'(z) - 1\right| \geqslant 1 - 1/K,$$

and then, $1 \leqslant L_f(z) = \frac{1}{2}(|F'(z) + 1| + |F'(z) - 1|) \leqslant K$.

So we have $l_f(z) \geqslant L_f(z)/K \geqslant 1/K$. Thus, we find

$$1/K \leqslant l_f(z) \leqslant L_f(z) \leqslant K. \tag{4.1}$$

Since $\lambda(f(z)) = \lambda(z)$, $z \in \mathbb{H}$, using (4.1) and the corresponding versions of 3A and 3B for $\mathbb{H}$, we obtain

$$\frac{1-k}{1+k} d_h(z_1, z_2) \leqslant d_h\big(f(z_1), f(z_2)\big) \leqslant \frac{1+k}{1-k} d_h(z_1, z_2).$$

It also follows from (4.1) that

$$\frac{1}{K}|z_2 - z_1| \leqslant \left|f(z_2) - f(z_1)\right| \leqslant K|z_2 - z_1|, \quad z_1, z_2 \in \mathbb{H}.$$

We leave to the reader to prove this inequality as an exercise.

This estimate is sharp (see also [4] for an estimate with some constant $c(K)$).

5. Schwarz lemma for hyperbolic harmonic and quasiconformal maps

We first need some properties of harmonic mappings.

5.1. Properties of harmonic maps

Let R and S be two surfaces. Let $\sigma(z)|dz|^2$ and $\rho(w)|dw|^2$ be metrics with respect to the isothermal coordinate charts on R and S, respectively, and let f be a C^2-map from R to S.

We use the following notation:

$$\mu = Belt[f] = \frac{q}{p}, \quad |\partial f|^2 = \frac{\rho}{\sigma}|f_z|^2, \quad |\bar\partial f|^2 = \frac{\rho}{\sigma}|f_{\bar z}|^2, \quad J(f) = |\partial f|^2 - |\bar\partial f|^2,$$

and the *Bochner* formula (see [10])

$$\mathbf{\Delta} \ln|\partial f| = -K_S J(f) + K_R. \tag{5.1}$$

Let us briefly explain how we apply the *Bochner* formula: Let f be a ρ-harmonic mapping, $\rho^* = \rho_f = \rho \circ f|p|^2$ and $K^* = K_{\rho^*}$ the Gaussian curvature of ρ^*. Recall, if σ is the euclidean metric density (that is $\sigma = 1$), it follows from the *Bochner* formula (5.1) that $K^* = K_S(1 - |\mu|^2)$.

Note that the *Bochner* formula is useful tool in this section (for ρ-harmonic mappings if the Gaussian curvature of ρ is negative), but it does not give new information for euclidean harmonic mappings.

Namely, if σ and ρ are euclidean metrics densities (that is $\sigma = \rho = 1$), then f is euclidean harmonic and application of the Bochner formula yields $\mathbf{\Delta} \ln|\partial f| = 0$. Also, this is an easy consequence of the fact that ∂f is an analytic function.

5.2. Ahlfors–Schwarz lemma for ρ-harmonic quasiregular maps

Although Wan did not state his result in the form of an inequality, his approach can be used to get the following result:

Theorem C *(Hyperbolic-qch version). Let f be a k-quasiconformal harmonic mapping from the unit disc $\mathbb{U}$ onto itself with respect to the Poincaré metric. Then for any two points z_1 and z_2 in U we have*

$$(1-k)d_h(z_1,z_2) \leqslant d_h\big(f(z_1), f(z_2)\big) \leqslant \sqrt{\frac{1+k}{1-k}}\, d_h(z_1,z_2), \tag{5.2}$$

where d_h is the hyperbolic distance induced by the hyperbolic metric in $\mathbb{U}$.

We now consider a generalization of Theorem C. In this paper we are actually concerned with a generalization of the right inequality in (5.2) and we postpone a more general discussion to a forthcoming paper.

For our purpose it is convenient to have the following lemma.

Lemma 1. *Let σ and ρ be two metric densities on $\mathbb{U}$, which define the corresponding metrics $ds = \sigma(z)|dz|^2$ and $ds = \rho(w)|dw|^2$, and let $f:\mathbb{U}\to\mathbb{U}$ be a C^1-mapping. If $\rho(f(z))L_f^2(z) \leqslant c\sigma(z)$, $z\in\mathbb{U}$, then $d_\rho(f(z_2), f(z_1)) \leqslant \sqrt{c}\, d_\sigma(z_2,z_1)$, for all $z_1, z_2 \in \mathbb{U}$.*

The proof of this result, which is a generalization of 3A, is straightforward and it is left to the reader as an exercise.

A version of the following result was announced in [7].

Theorem 5.1 *(ρ-qrh version). Let R be a hyperbolic surface with the Poincaré metric density λ, S another with a metric density ρ and let the Gaussian curvature of the metric $ds^2 = \rho(w)|dw|^2$ be uniformly bounded from above on S by the negative constant $-a$, $a>0$. Then any ρ-harmonic k-quasiregular map f from R into S decreases distances up to a constant depending only on a and k:*

$$d_\rho\big(f(z_1), f(z_2)\big) \leqslant \frac{1}{\sqrt{a}}\sqrt{\frac{1+k}{1-k}}\, d_h(z_1,z_2), \tag{5.3}$$

where d_ρ is the corresponding distance induced by the metric $ds^2=\rho(w)|dw|^2$ on S.

Proof. By the uniformization theorem we can suppose that R and S are the unit discs.

Let $\rho^* = \rho\circ f|p|^2$, $\rho_0 = a(1-k^2)\rho^*$ and $K_0 = K(\rho_0)$ the Gaussian curvature of ρ_0. Set $K^* = K(\rho^*)$. First, we show that ρ_0 is an ultrahyperbolic metric density. Namely, if $\rho_0 \neq 0$ (that is $p(z_0) = f_z(z_0) \neq 0$), then there is a neighborhood W of z_0 such that f is one-to-one in W. Using the fact that $K^* = K_S(1-|\mu|^2)$, we conclude that $K^* \leqslant -a(1-k^2)$ and therefore $K_0 \leqslant -1$ on W. Thus, ρ_0 is an ultrahyperbolic metric on $\mathbb{U}$.

Hence, by the Ahlfors ultrahyperbolic lemma,

$$a\big(1-k^2\big)\rho^* \leqslant \lambda \quad \text{and} \quad a\big(1-k^2\big)\rho\big(f(z)\big)L_f^2(z) \leqslant K^2\lambda, \quad z\in\mathbb{U}.$$

An application of Lemma 1 immediately yields the result. □

Note that one can show that there is a qc mapping g and an analytic function F such that $f = F \circ g$.

Using the uniformization theorem, some results of this paper can be extended to a more general setting including Riemann surfaces, more general functions and metrics on both domains and codomains.

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